Firm-bank linkages and optimal policies in a lockdown

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Abstract

We develop a novel framework featuring loss amplification through firm-bank linkages. We use it to study optimal government support in a lockdown that creates heterogeneous revenue losses to firms, which must borrow from banks. Firms’ increase in debt reduces their output due to moral hazard. Banks need safe collateral to raise funds. Without government support, aggregate risk constrains bank lending, amplifying output losses. Optimal support provides sufficient aggregate risk insurance, and is implemented with firm-specific transfers, fairly-priced guarantees on bank debt, and procyclical firms’ taxation to achieve a fiscal surplus target. Our results shed light on suboptimality features in the actual policy responses.

JEL Classification: G01, G20, G28

Keywords: Covid-19, lockdown, firms’ debt, moral hazard, bank equity, aggregate risk, government policies

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1 Introduction

The lockdown measures introduced to contain the diffusion of Covid-19 led to significant cash-flow shortages for businesses. To prevent an immediate wave of corporate defaults, governments implemented a variety of policies to help firms cope with their liquidity needs. The most common policies were transfers to firms, which directly cover their liquidity needs, and guarantees on new bank loans, which indirectly help firms by supporting bank lending.\(^1\) During the initial stages of the pandemic crisis, international institutions warned of increasing risks of negative macro-financial feedback loops similar to those prevalent in the Global Financial Crisis (GFC) (IMF [2020a], FSB [2020], ESRB [2021]). The main concerns at the time were widespread debt overhang problems for firms (Brunnermeier and Krishnamurthy [2020], Crouzet and Tourre [2021]), and their impact on banks’ credit portfolio (Blank et al. [2020], Acharya et al. [2021]).

Government interventions have (so far) avoided the materialization of those risks, and the attention has now turned to the implications for public debt sustainability of the unprecedented public intervention, whose size reached up to 40% of GDP for some advanced economies (IMF [2021b], Kose et al. [2021]). The actual bill for some governments could be much higher given the contingent liabilities implied by the interventions (see IMF [2020b], IMF [2020c]). We ask, what is the optimal way to support firms during lockdowns? What role do guarantees channeled through banks play? How should government interventions be financed? Answering these questions is important for two reasons. First, although the initial interventions have been able to avoid a wave of firms’ defaults, our analysis allows to understand whether that objective could have been achieved at a cheaper cost for the taxpayer. Second, in case additional support to firms were necessary to deal with new waves of the pandemics, our results help to guide its design in a context in which the balance sheets of firms, banks, and fiscal authorities are likely to be weaker.

The paper develops a theoretical framework of bank intermediation and real activity that considers financing frictions both at the firm and the bank level. Our focus is on small and

\(^1\)Additional support to firms was provided by central banks through purchases of corporate debt and commercial paper in primary markets and funding for lending programs intermediated by banks. Supervisory authorities also aimed at favoring firms’ access to new lending through reductions in capital requirements and temporary waivers of loan classification rules.
medium enterprises (SMEs), which are dependent on bank funding and were severely hit by the Covid-19 crisis (see Gourinchas et al. [2021]). The model is used to study optimal policy design in a lockdown that creates heterogenous revenue losses to firms, which as a result need funding from banks in order to survive. The framework features a feedback mechanism that amplifies initial output losses through the balance sheet linkages of banks and firms. We consider a government whose debt expansions above a fiscal slack threshold create deadweight costs, and show that optimal government support can be implemented with a combination of i) firm-specific transfers that partially neutralize heterogeneity across firms in the lockdown shock; ii) fairly priced guarantees on banks’ debt that provide aggregate risk insurance which helps alleviating bank financing frictions, and iii) a procyclical future taxation of firms’ profits necessary to stabilize government’s debt levels due to the contingent liabilities created by guarantees.

The optimality to support firms through these policies contributes to our understanding of government interventions during crises. Following the GFC, a body of research on the amplification role of financial frictions has highlighted the importance of transfers to repair the balance sheet of borrowers following negative shocks (see for instance Gertler and Kiyotaki [2010], He and Krishnamurthy [2013], or Brunnermeier and Sannikov [2014]). Most of this literature considers a single financially constrained sector, typically interpreted as a banking sector that owns and manages productive firms, and hence cannot address the question of the relative effectiveness of direct and indirect support raised by the policy response to the pandemic. A few recent dynamic macroeconomic models exhibit firm-bank linkages (e.g., Rampini and Viswanathan [2019], Elenev et al. [2021], Elenev et al. [2022], and Villacorta [2020]), but do not formally address optimal policy design in response to a shock to the corporate sector, which is the focus of our more stylized model. Moreover, our framework includes a government whose debt expansions entail real costs, endogenously giving rise, through support policies, to balance sheet linkages between firms, banks and the government. While in

\footnote{SMEs’ vulnerabilities were a primary concern for governments around the world as evidenced by the more than 500 government programs targeting them (see the Financial Response Tracker at https://som.yale.edu). We discuss in Section 6.3 how our framework can be adapted to analyze some of the joint interventions of central banks and fiscal authorities to support larger firms’ access to debt markets during the Covid-19 crisis.}

\footnote{There is ample evidence that the impact of the Covid-19 crisis on small firms has been highly heterogeneous along different dimensions such as sector of activity, region, owners’ gender or race, and possibility to switch to online work (Fairlie [2020], Bloom et al. [2021] for the US, Fernández Cerezo et al. [2021] for Spain, or Puy and Rawdanowicz [2021] for OECD countries).}
the aftermath of the GFC the attention of the policy debate focused on the sovereign-bank nexus (see Dell’Ariccia et al. [2018]), as economies walk the exit path from the Covid-19 crisis amidst high corporate and public indebtedness the focus of the policy attention is on an enlarged sovereign-bank-corporate nexus (see Schnabel [2021]). Our paper provides to the best of our knowledge the first framework able to capture such three-party nexus and derive optimal policy design implications.

We consider a competitive model of bank intermediation in which the initial firm specific revenue losses created by a lockdown get amplified through firm-bank linkages due to two frictions. First, an increase in firms’ debt reduces firms’ output and the value of their outstanding debt. This is because entrepreneurs are subject to moral hazard when they raise external funds. Second, banks need safe collateral to raise funds. This is because investors with available funds have an absolute preference for safety. The diversification of firms’ idiosyncratic risks allows banks to issue some safe debt, but aggregate risk limits banks’ ability to pledge their assets in order to raise funding.

At the heart of the novel loss amplification mechanism in our model is a tension between firms and banks’ equity, that is, the expected value of their residual claims. In order to survive, firms need to promise some of their future payoffs to obtain enough bank funding. Banks in turn can only pledge the safe part of those payoffs as collateral to borrow from investors. In equilibrium, banks’ equity must increase to provide a buffer against losses that protects their new debt investors. The increase in banks’ equity requires a larger distribution of payoffs away from firms to banks, which creates lending rents and reduces firms’ equity. The endogenous redistribution of equities across the two sectors is not welfare neutral: by aggravating firms’ moral hazard, it further amplifies output losses. For firms most severely hit by the lockdown, the interplay of frictions at the firm and bank level freezes new financing flows, and leads to

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4The reduction in firm value due to increases in leverage would also result from contractual frictions that give rise to debt overhang problems à la Myers [1977].

5This bank funding friction, which in particular implies that banks cannot issue equity, aims at capturing that during episodes of high economic uncertainty investors tend to have a strong preference for high quality riskless assets. While we emphasize this market driven borrowing constraint, bank leverage and lending could also be limited by aggregate risk due to regulatory requirements.

6There is evidence that firms in the US partially covered their liquidity needs by drawing their pre-committed bank credit lines (Acharya and Steffen [2020], Li et al. [2020]). Our results would be robust to the inclusion of credit lines with a fixed interest rate provided firms’ funding needs during the lockdown exceed their pre-committed amount of credit.
their liquidation even when their NPV would be positive absent frictions.

We consider a government that sets policies in order to contain the amplification of the initial lockdown losses and assume there are convex deadweight costs when interventions are financed by expanding public debt. Our initial set of results provides the main properties of optimal support policies. First, we show that firms with the largest lockdown losses must be liquidated, while the remaining firms should obtain funding and continue with a debt burden that is higher than before the lockdown but constant across them. The optimal intervention thus redistributes some net worth from lowly hit firms, whose moral hazard problem is mild, to mediumly hit firms, for which it is more severe. Importantly, most severely hit firms are left outside the intervention perimeter because their balance sheet “repair” would require too many resources.

Second, we find that optimal government policies must protect against aggregate shocks to the infinitely risk averse investors that provide their funds in the lockdown. The aggregate risk insurance provided by support policies substitutes for the loss absorption role of bank’s equity, ensuring that banks do not obtain rents from their intermediation between investors and firms. By preventing that equity values get redistributed from firms to banks, firms’ skin-in-the-game and output are maximized.

Policies can provide aggregate risk insurance to investors in low future states of the world either by distributing resources away from banks and firms or by expanding public debt. Since the latter entails (convex) costs, the final property of optimal policies states that aggregate risk insurance must be first provided through the taxation of firms and/or banks, and only when their taxable profits are exhausted, a recourse to public debt expansion should be done. Such “pecking order” in aggregate risk insurance provision minimizes the volatility of the public debt level to the extent possible. We show that for governments’ with large fiscal slack (which we model as a public debt level threshold above which further debt expansions create strictly positive deadweight costs), the future public debt level is constant under the optimal policy, while for governments’ with a low fiscal slack it exhibits some volatility. This is because, when the government has a small budget, investors must provide a larger fraction of the firms’ financing during the lockdown and more aggregate risk insurance is needed. Yet, the ability of the government to provide it through the taxation of firms and banks is limited, and it has to be complemented with some public debt expansion upon the worst shocks. So, on top of
creating volatility in public debt, investors’ demand for safety also distorts (relative to a risk-neutral environment) the intensive and extensive margins of optimal support to firms: more firms are liquidated, which reduces the need of investors’ funds and the associated required aggregate risk insurance, while the firms that continue exhibit a lower overall debt burden, which increases their skin-in-the-game and the output that can be taxed upon worst shocks.

We then move to analyze how optimal interventions can be implemented in a decentralized competitive environment. We show this can be achieved through the combination of three policies, each of them addressing one of the optimality properties above described.

The first policy consists of government firm-specific transfers which are funded through an initial public debt expansion. The policy covers only firms that do not experience a too severe revenue shock. The size of the transfers increases with the firms’ revenue loss (and is negative, that is, a tax, for the lowest hit firms), thereby neutralizing heterogeneity in the lockdown shock among these firms. Hence, firms covered by the policy continue with the same level of debt (which is higher than before the lockdown), while severely hit firms do not get transfers and are liquidated.

The second policy is a guarantee on banks’ debt on which banks pay a fairly priced fee in the future. These guarantees lead to transfers from the government to the bank upon bad aggregate shocks, in which the guarantee is executed, and from the bank to the government upon good aggregate shocks, in which the fee is reimbursed. They thus provide aggregate risk insurance to the banks, relaxing their funding constraints and making their lending as cheap as possible. As banks’ debt guarantees eliminate scarcity rents associated with the aggregate loss absorption role of banks’ equity, firms’ skin-in-the-game and output increase.

The third policy is a fiscal surplus target that the government pursues to stabilize its debt level in the post-lockdown dates, through the taxation and subsidization of firms’ future profits. Given the contingent liabilities associated with banks’ debt guarantees materialize upon bad aggregate shocks, the fiscal surplus target requires the government to tax firms’ profits in those contingencies. Instead, upon good aggregate shocks in which banks make profits and reimburse the guarantee fee, the government uses the excess of fiscal surplus to subsidize firms’ profits. The optimality of a procyclical future firms’ taxation to finance the support to firms during the lockdown can be interpreted as one of the fiscal “legacies” of the pandemic.
crisis. The taxation and subsidization policy is anticipated by firms but is designed to be overall neutral on their effort incentives.\(^7\)

The actual government policies in support to firms during the Covid-19 crisis include some of the tools in our model’s optimal mix, or use similar ones. Transfers to firms indexed to the economic impact of lockdowns have been widely used. New firm financing has been frequently supported through public guarantees on bank loans, typically with no fees attached. Within our model these guarantees would constitute an alternative way to provide aggregate risk insurance in the economy. In contrast with our optimal fiscal policy prescriptions, government interventions have not come along with credible fiscal consolidation plans, whose implementation might be difficult due to technical, practical and political obstacles (Kose et al. [2021]). We finally analyze the welfare implications of a government’s inability to set its future fiscal policy and show that in such suboptimal environment the issuance of fairly priced bank debt guarantees would be preferable to that of bank loan guarantees. The reason is that, due to firms’ idiosyncratic risk, bank loan guarantees lead to government disbursements upon all aggregate contingencies, while bank debt guarantees imply disbursements only upon the worst shocks and of the minimum magnitude that makes debt safe, reducing public debt volatility.

**Related literature.** From a modeling perspective, our paper is mostly related to Holmstrom and Tirole [1998]. That paper focuses on the ex-ante design of contracts for liquidity provision when firms anticipate the possibility of liquidity shocks and, due to moral hazard problems, face constraints on their ex-post external financing capacity. We focus instead on an aggregate unexpected liquidity shock and the ex-post liquidity provision given existing firms and banks’ legacy debt. We assume in addition that banks are funded with safe debt, which limits their supply of lending to firms, aggravating the firms’ moral hazard problem. The interplay between these two frictions gives rise to amplification mechanisms affecting policy design that are absent in Holmstrom and Tirole [1998].

Our paper is also related to theoretical contributions in which frictions give rise to external financing constraints for both banks and firms. Holmstrom and Tirole [1997], Repullo and Suarez [2000], and Rampini and Viswanathan [2019] highlight how shocks to the net worth

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\(^7\)As described in the third optimality property above, fiscally weak governments will not be able to achieve their fiscal surplus target upon the worst shocks even with maximum taxation of firms’ profits, so that their debt level will exhibit some volatility.
of one of the set of agents gets amplified due to balance sheet linkages, but do not consider optimal intervention design, which is the focus of our paper. The optimality of government transfers to firms or banks during crises is analyzed in Villacorta [2020], which shows in a dynamic macroeconomic model that the optimal transfer target depends on how negative shocks affect the distribution of net worth between banks and firms. In the related model with default costs in Allen et al. [2015], all the risk in the economy is non-diversifiable, so that banks and firms’ defaults are perfectly correlated and they find socially optimal that all the equity in the economy is allocated to firms. The optimal policy response in a crisis in such set-up (something the paper does not address) would only consist of direct transfers to firms, and policies that support banks’ capability to intermediate by diversifying idiosyncratic firms’ risks would play no role.

Our paper belongs to the growing literature that analyzes optimal interventions by fiscal or monetary authorities during the Covid-19 crisis. Part of the literature has focused on the role of fiscal and monetary policy interventions in macroeconomic models in which the lockdown gives rise to supply shocks that get amplified through demand factors (Guerrieri et al. [2022] and Caballero and Simsek [2021]), or creates falls in demand in some sectors which could potentially propagate to other sectors (Faria-e Castro [2021] and Bigio et al. [2020]). Our paper abstracts from aggregate demand factors and instead focuses on shock amplifications stemming from balance sheet linkages between firms and banks. Regarding the focus on support policies to firms, the closest paper to ours is Elenev et al. [2022], which builds on the dynamic macroeconomic framework with constrained firms and banks developed in Elenev et al. [2021], and assesses quantitatively the effectiveness of the different corporate relief programs introduced in the US. The paper finds that forgivable bridge loans, which simulate the Paycheck Protection Program and could be interpreted as direct transfers in our model, are more effective than purchases of risky corporate debt, which simulate the Corporate Credit Facilities, and partial bank loan guarantees, which simulate the Main Street Lending Program. Our paper contributes to these findings by highlighting the optimality of introducing new

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8 In this respect, our paper is also related to Arping et al. [2010], which analyze the optimality of supporting firms’ investment through co-funding or loan guarantees in a set-up in which banks’ net worth is not relevant.

9 A strand of papers has focused instead on the optimal health policy response given the interaction between the evolution of the pandemic and the macroeconomy (for example, Eichenbaum et al. [2021], Alvarez et al. [2021], Acemoglu et al. [2021], Jones et al. [2021], Correia et al. [2020]).
policies that provide aggregate risk insurance and are fairly reimbursed in the future, such as the issuance of fairly priced bank debt guarantees.

The rest of the paper is organized as follows. Section 2 describes the model set-up. Section 3 analyzes the equilibrium and highlights how lockdown losses are amplified through firm-bank linkages in absence of policy support. Section 4 analyzes the general optimal policy design problem and how it depends on the fiscal strength of the government. Section 5 shows that a policy toolkit consisting of firm-specific transfers and taxes at the initial date, fairly priced public guarantees on banks’ debt and procyclical taxation of firms’ profits in the future is able to achieve optimality. Section 6 discusses the actually observed policy responses during the Covid-19 lockdowns, analyzes the implications of the lack of capability to set a future fiscal policy, and adapts the model to interpret interventions by central banks and fiscal authorities in corporate debt markets. Section 7 concludes. The proofs of the formal results in the paper and some formal details on issues not covered in the main text can be found in the Appendix.

2 Model set-up

Consider an economy with two dates, \( t = 0, 1 \), and four types of agents with a zero discount rate: investors, firms, competitive banks that intermediate funds between investors and firms, and a government. Investors are infinitely risk averse, the remaining agents are risk-neutral. Both firms and banks have at the initial date some assets and liabilities, and the government can obtain external resources at \( t = 1 \) at a weakly increasing marginal deadweight cost.

**Investors.** Investors have deep pockets. We follow Gennaioli et al. [2013] and model investors’ infinite risk-aversion by assuming that they derive linear utility from consumption at \( t = 1 \) at their worst-case scenario.\(^{10}\) We hence assume that they only invest in riskless assets. In reality, the agents we refer to as investors might be either wealthy private agents that during stressful and highly uncertain periods have the capability to rapidly deploy liquidity but only against riskless collateral, or central banks whose mandate imposes limitations on the loss exposure associated with their interventions.

\(^{10}\)For a given set \( \Omega \) of states of nature at \( t = 1 \), Gennaioli et al. [2013] define the utility \( U \) derived by an infinitely risk-averse agent from a stochastic consumption distribution \( (c_1(\omega))_{\omega \in \Omega} \) at \( t = 1 \) as \( \bar{U} \equiv \min_{\omega \in \Omega} c_0 + c_1(\omega) \).
Firms. At $t = 0$, there is a continuum of firms with measure one, each firm has a risky project in place and old debt with promise $b_0 > 0$ due at $t = 1$. Firms’ debt is held by the banks that are described below. In order to continue, firms have to incur operating costs at $t = 0$ that, in absence of a lockdown, would be paid out of their projects revenues at $t = 0$.

There is an economic lockdown at $t = 0$ that destroys some of those revenues. Each firm faces as a result a cash-flow shortfall $\rho \in [0, \bar{\rho}]$ at $t = 0$, where $\bar{\rho} > 0$, and needs to raise new funding to continue. We refer to $\rho$ as the firm-type and assume it follows a continuum distribution with density $h(\rho) > 0$ for all $\rho \in [0, \bar{\rho}]$. The cumulative distribution function is denoted $H(\rho)$.

If a firm continues, its pay-off at $t = 1$ is $A > 0$ in case of success, and zero in case of failure. Firms’ success at $t = 1$ depends on idiosyncratic and aggregate shocks as described later. The probability as of $t = 0$ that a firm succeeds is denoted with $p$ and satisfies $p \in [0, \bar{p}]$, where $\bar{p} < 1$. The success probability $p$ coincides with the unobservable effort exerted by the firm at $t = 0$, which entails a disutility described by a function $e(p)$ satisfying:

**Assumption 1.** $e(0) = 0, e'(0) = 0, e'(\bar{\rho}) = A - \eta$, for some $\eta \in (0, A)$, $e''(0) > 0, e''(\bar{\rho}) > \eta / \bar{p}$ and $e'''(p) \geq 0$ for all $p$.

The assumption implies that the expected firm pay-off net of effort cost, which we refer to as expected output, is maximized at the upper bound on effort, that is:

$$\bar{p} = \arg \max_{p \in [0, \bar{p}]} \{ pA - e(p) \}. \quad (1)$$

Firms’ moral hazard. The unobservability of the firm’s effort creates a moral hazard problem: the firm chooses effort in order to maximize the value of its profits, not its expected output. Specifically, for a general debt promise $b \in [0, A]$ at $t = 1$, the firm’s optimal effort choice, denoted $\hat{p}(b)$, satisfies:

$$\hat{p}(b) = \arg \max_{p \in [0, \bar{p}]} \{ p (A - b) - e(p) \}. \quad (2)$$

The next result follows from Assumption 1.

**Lemma 1.** For a debt promise $b \in [0, A]$, the optimal firm effort $\hat{p}(b)$ satisfies:

- If $b \in [0, \eta]$, then $\hat{p}(b) = \bar{p}$.
- If $b \in (\eta, A]$, then $\hat{p}(b) < \bar{p}$, $\hat{p}(b)$ is given by
\[ e'(\hat{p}(b)) = A - b, \]  

and satisfies 
\[ \frac{d\hat{p}(b)}{db} < 0, \quad \frac{d[\hat{p}(b)A - e(\hat{p}(b))]}{db} < 0. \]

In addition, there exists \( b_{\text{max}} \in (\eta, A) \) such that 
\[ \frac{d[\hat{p}(b)b]}{db} > 0 \iff b < b_{\text{max}}. \]

The results and intuitions in the lemma are as follows. The presence of a debt promise \( b \) creates a moral hazard problem on firms’ effort because part of the value from this action is appropriated by debtholders. Notwithstanding this, if the firm’s debt promise is small \((b \leq \eta)\), the effort choice remains at its maximum and efficient level. When the debt promise is instead large \((b > \eta)\), effort becomes inefficiently low and the firm’s expected output falls. Despite the reduction in effort, the expected value of the debt promise \( b \), amounting to \( \hat{p}(b)b \), increases provided \( b \) is not too large \((b < b_{\text{max}})\).

We focus to fix our ideas in a situation in which, under no lockdown, the firms’ old debt does not create inefficiencies in effort:

**Assumption 2.** \( b_O = \eta \).

Finally, a firm that does not continue is liquidated at \( t = 0 \), which yields a recovery value \( R \) that we assume is realized at \( t = 1 \). We make two assumptions:

**Assumption 3.** \( R = \overline{p}b_O \).

This assumption is done for simplicity and implies that the expected value of the old debt equals the recovery value of the firm.

**Assumption 4.** \( \overline{p} = \overline{p}A - e(\overline{p}) - R \).

This assumption implies that the continuation of the firms that exhibit the largest cash-flow shortfall in the lockdown is efficient if and only if they exert maximum effort, which from Lemma 1 is the case under the old promise \( b_O \) but not under a promise \( b > b_O \).\(^{11}\)

\(^{11}\)Notice that Assumptions 2 and 3 imply that \( \overline{p}A - e(\overline{p}) - R = \overline{p}(A - b_O) - e(\overline{p}) \geq \rho \), for all \( \rho \leq \overline{p} \), so that firms would optimally pay for the operating costs if they had the funds.
**Banks.** At $t = 0$, there is a measure $\rho$ of banks indexed by $\rho \in [0, \bar{\rho}]$. Bank $\rho$ is specialized in lending to $\rho$-type firms. At the initial date, it holds the old firm debt promises $b_0$ of the mass $h(\rho)$ of $\rho$-type firms and is funded with $d_0 h(\rho)$ units of debt due at $t = 1$ and held by investors, where $d_0$ is constant across banks. Hence, in absence of a lockdown, the capital structure of all banks and the success probability of their loans would be the same.

Firms need financing from their banks at $t = 0$ to continue. We assume that $d_0 < R$, and we have from Assumption 3 that $R < b_0$, so that upon liquidation of $\rho$-type firms the entire recovery value $R$ of the firms at $t = 1$ would be seized by their bank, which in turn would be able to repay in full its old debt $d_0$ and appropriate the residual $R - d_0$. Firms thus find optimal to demand new financing from their banks to cover their cash-flow shortfall.

New financing is intermediated as follows: bank $\rho$ provides $\rho$ units of funds to each $\rho$-type firm at $t = 0$ against a new overall debt promise $b_N$ at $t = 1$, and raises the necessary funds by issuing new safe debt to investors. The bank is able to intermediate between investors, who demand safe assets, and firms, whose output at $t = 1$ is risky, because it takes advantage of the diversification opportunities in the economy, which we describe next.

At $t = 1$, an aggregate shock $\theta$ that affects the pay-off of all firms that continue is realized. Specifically, if $\rho$-type firms’ effort is denoted with $p$, their success probability conditional on the realization of $\theta$ is $\theta p$. Hence, when $\theta > 1$ ($\theta < 1$) the conditional success probability is larger (lower) than its unconditional value. The aggregate shock satisfies $E[\theta] = 1$ and its support is $[\theta, 1/\bar{\rho}]$, with $\theta \in (0, 1)$. We assume that, conditional on the aggregate shock $\theta$, $\rho$-type firms’ pay-offs are independent, so that by the law of large numbers a deterministic share $\theta p$ of them succeed.

Under these assumptions, the $\theta$-contingent return at $t = 1$ of bank $\rho$ portfolio of new firms’ debt promises $b_N$ is $\theta p b_N h(\rho)$. Its minimum is achieved for the worst aggregate shock, $\theta = \theta$, and amounts to $\theta p b_N h(\rho) > 0$. Crucially, while the lowest pay-off at $t = 1$ of the debt issued by each firm is zero, the lowest pay-off of the bank’s debt portfolio is strictly positive. The diversification of idiosyncratic firm risk thus allows the bank to issue safe debt.

**The competitive bank lending contract.** We next describe how the new debt promise $b_N$ for the financing of $\rho$-type firms during the lockdown is determined. We say that $b_N$ is *feasible* if two conditions are satisfied:
• **Bank funding constraint:** bank $\rho$ is able to issue the required safe debt while the old debt remains safe which, after normalizing by the measure $h(\rho)$ of $\rho$-type firms, can be written as:

$$d_O + \rho \leq \hat{\Theta}(b_N) b_N,$$

where the RHS captures the lowest payoff of the bank’s loan portfolio, and takes into account that firms’ effort given $b_N$ is $p = \hat{p}(b_N)$ (Lemma 1). Notice that we require old debt to remain safe. This could result from its seniority, or, interpreting it as short-term debt that has to be rolled-over, from the possibility that existing debtholders run on the bank and force the firms’ liquidation when they fear not to be repaid in full at $t = 1$. Let us highlight that the funding constraint in (4) can be equivalently interpreted as a maximum leverage constraint imposed by investors.

• **Minimum bank profit constraint:** bank $\rho$ prefers to provide the new financing to firms rather than to liquidate them:

$$\Pi(b_N) \equiv E \left[ (\theta \hat{\rho}(b_N) b_N - d_O - \rho)^+ \right] \geq R - d_O,$$

where the expression for the bank’s expected profits in the LHS accounts for the effect of $b_N$ on firms’ effort.

Whenever some feasible promise exists, we assume that the bank chooses the minimum one, which we denote as $b'_N(\rho)$ and refer to as the *competitive loan promise* for the lockdown financing. Such promise maximizes the firms’ profits and would arise as a result of Bertrand competition on the new financing to firms between the bank and a potential new bank entrant (see Appendix A.1 for details). In addition, we have immediately from Assumption 3 that for any firm-type $\rho \geq 0$, the competitive promise $b'_N(\rho)$ (whenever it exists) satisfies $b'_N(\rho) \geq b_O$, with equality if and only if $\rho = 0$.

Finally, we assume that:

**Assumption 5.** $d_O = \overline{\rho} b_O$.

The assumption states that the banks have at $t = 0$ no capital “buffer” as their old debt $d_O$.

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12We will henceforth make analogous normalizations in banks’ balance sheet variables without explicit mention.
equals the safe value of their old loan portfolio $b_0$.\footnote{All our results on optimal policy design would also hold when banks have at $t = 0$ a positive capital buffer, that is when $\theta pb_0 - d_0 > 0$, provided such buffer is not too large.}

**Government.** We consider a government that aims at maximizing welfare in the economy. The government can tax and subsidize private agents. In addition, we assume in a reduced form manner that the government can obtain $x$ units of external resources at $t = 1$ and distribute them to private agents, which creates government (dis)utility $u_G(x)$ given by:

$$u_G(x) = -x - g((x - X)^+),$$

where $X > 0$, and the function $g(.)$ satisfies:

**Assumption 6.** $g(0) = g'(0) = 0$, $g''(y) > 0$ for $y \geq 0$.

These assumptions allow to interpret the variable $X$ as the government’s fiscal slack: the redistribution of external resources $x$ from the government to private agents below $X$ has a zero net welfare effect, but such effect is negative when the redistribution exceeds $X$. For the sake of concreteness, we will henceforth assume that $x$ is obtained by issuing public debt that is repaid at some unmodeled future date, and will interpret the social deadweight costs captured by $g((x - X)^+)$ as resulting from the need to cut welfare programs in the future when public debt becomes too large, or from the inflation costs born by future generations stemming from the the monetization by a central bank of large public debt expansions.\footnote{See Kose et al. [2021] for a description of the potential costs that countries may face when dealing with excessive government debt levels.}

We assume that:

**Assumption 7.** $X < E[\rho]$.

The assumption implies that the government’s fiscal slack is not sufficient to totally cover the overall firms’ cash-flow shortfall during the lockdown.

## 3 Benchmark: No intervention and firm-bank linkages

In this Section, we characterize the competitive bank financing to firms in the lockdown in the absence of government intervention, and highlight how firms’ moral hazard in effort interacts with the bank funding constraint, generating firm-bank linkages that amplify the initial lockdown losses.
Consider some \( \rho \in [0, \bar{\rho}] \) and suppose there is a feasible loan promise for the financing of \( \rho \)-type firms, that is, a promise \( b_N \) that satisfies bank \( \rho \) funding and minimum profit constraints in (4) and (5), respectively. We can use Assumption 5 to rewrite the constraint (4) as:

\[
\rho \leq \bar{\theta} \hat{p}(b_N) b_N - \bar{\theta} \hat{p}(b_O) b_O, \tag{7}
\]

which states that the new debt \( \rho \) the bank issues to finance firms have to be backed by the increase in the overall safe collateral value of its loans. From Lemma 1, we have that even though \( \hat{p}(b_N) \) is decreasing in \( b_N \) for \( b_N \geq b_O = \eta \), the safe collateral value of the bank loans, \( \bar{\theta} \hat{p}(b_N) b_N \), is increasing in \( b_N \) for \( b_N < b_{\text{max}} \), which explains why the bank has some capability (albeit limited) to issue safe debt despite the firms’ moral hazard.

Let \( b_N' = b_N'(\rho) \) denote the competitive promise for the financing of \( \rho \)-type firms. It is easy to prove that \( b_N' \in [b_O, b_{\text{max}}] \) and at least one of the bank constraints is binding at \( b_N' \).\(^{15}\) Suppose that the funding constraint in (4) (or (7)) is binding:

\[
\rho = \bar{\theta} \hat{p}(b_N') b_N' - \bar{\theta} \hat{p}(b_O) b_O. \tag{8}
\]

We have that the bank’s profits after granting new financing to the firms, whose expression is in the LHS of (5), can be rewritten as:

\[
\Pi(b_N') = (d_O + \rho) \frac{1 - \theta}{\bar{\theta}}. \tag{9}
\]

The bank’s profits thus amount to the product of its debt, \( d_O + \rho \), and a term that captures the rents the bank obtains per unit of debt, \( (1 - \theta)/\bar{\theta} \). Using Assumptions 3 and 5, we have from (9) that

\[
\Pi(b_N') = R - d_O + \rho \frac{1 - \theta}{\bar{\theta}} > R - d_O, \tag{10}
\]

and the bank minimum profit constraint in (5) is strictly satisfied. Despite the bank being competitive, its profits increase from the new lending to firms because the funding constraint faced by the bank limits its capability to provide cheap lending (that is, to set a low new promise \( b_N \)). Intuitively, upon bad shocks many firms default on their loans and the new loan promise \( b_N \) must be large enough for the repayment of successful firms to be able to repay the old and new bank debt in full. The larger the amount of debt that need to be repaid in full upon bad shocks, the larger the profits the bank makes upon good shocks in which more firms are successful, and hence the larger the expected bank profits.

To gain more intuition on how the firms’ moral hazard and the bank’s funding frictions

\(^{15}\)The reason is that for \( b_N \in [b_O, b_{\text{max}}] \) both (4) and (5) get relaxed as \( b_N \) increases.
affect the funding cost of firms, consider a bank that were able to raise funds through the issuance of risky debt or equity to some unmodeled risk-neutral investors at a required expected rate of return equal to one. Such bank would not be subject to the funding constraint in (4) and would be able to provide new financing to the firms in exchange of a competitive promise \( \tilde{b}_N \)' that makes (5) binding, which using Assumption 3 can be rewritten as:

\[
\rho = \hat{p}(\tilde{b}_N') \tilde{b}_N' - \hat{p}(b_O) b_O,
\]

which states that the increase in the value of the competitive bank’s liabilities and assets are equal. We have from (8) and (11) that \( \tilde{b}_N' < b_N' \) if \( \rho > 0 \), so that the presence of the bank funding constraint increases the firms’ funding cost. The measure of such increase is given by the difference:

\[
\hat{p}(b_N') b_N' - \hat{p}(\tilde{b}_N') \tilde{b}_N' = \rho \frac{1 - \theta}{\theta} + \left[ \hat{p}(\tilde{b}_N') - \hat{p}(b_N') \right] b_O,
\]

which we have decomposed as the sum of two terms using (8) and (11). The first one captures the increase in bank profits (bank equity) required to provide enough loss absorption capacity to back safe debt issuance. The second one captures that those rents from safe collateral creation aggravate the firms’ moral hazard problem, leading to additional losses on the bank’s existing loans that the bank must cover by setting an even higher promise on new financing.

The increase in the firms’ funding cost due to the bank’s funding constraint not only reduces firms’ equity to increase bank equity: it also reduces overall firm value. In fact, taking into account that \( b_O < \tilde{b}_N < b_N' \leq b_{\text{max}} \) if \( \rho > 0 \), we have from Lemma 1 the following comparison of firms’ value (net expected output):

\[
\hat{p}(b_N') A - e(\hat{p}(b_N')) - \rho < \hat{p}(\tilde{b}_N') A - e(\hat{p}(\tilde{b}_N')) - \rho < [\hat{p}A - e(\hat{p})] - \rho,
\]

which effectively show that the interaction of frictions at the firm and bank level creates balance sheet linkages that amplify the lockdown losses relative to those when there are only firm frictions (first inequality), which in turn exceed the direct losses \( \rho \) created by the lockdown (second inequality).

Finally, we illustrate in Figure 1 our discussion in this section. Full lines present variables in our baseline economy with frictions at the firm and bank level, and segmented lines present variables in an alternative economy with risk-neutral investors in which the bank is not subject to funding frictions. Panel 1a shows that, when there are bank frictions in addition to firm...
frictions, there is a profit redistribution from firms to banks, as banks need to increase their equity to back safe debt issuance. Panel 1b in turn illustrates that such equity redistribution amplifies initial firm value losses, which are presented in the dotted line for each type of firm. The two effects are increasing in the size of the cash-flow shortfall $\rho$. And, when $\rho$ exceeds a threshold, which is lower in presence of firm and bank frictions, firms are liquidated, generating a discontinuous drop in the profits of banks, firms and in firm value.\textsuperscript{16} Loss amplification through firm-bank linkages thus lead to the liquidation of more firms than when there are only firm frictions.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Effects from firm and bank frictions for given firms’ cash-flow shortfall}
\end{figure}

\textbf{Note:} The figure exhibits the effects from the presence of firm and bank frictions in the baseline economy (solid lines) relative to an alternative economy with risk neutral investors and only firm (moral hazard) frictions (segmented lines), for different values of the firms’ cash-flow shortfall $\rho$. Panel a): Firm and bank profits, $\tilde{p}(b_N)(A - b_N) - e(\tilde{p}(b_N)), \tilde{p}(b_N)b_N - d_O - \rho$, respectively, where $b_N \in \{b_N' (\rho), b_N(\rho)\}$ is the competitive promise in each economy. Panel b): firm value difference relative to no lockdown, $Y(b_N) - Y_0$, where $Y(b_N) = \tilde{p}(b_N)A - e(\tilde{p}(b_N)) - \rho$, $Y_0 = \tilde{p}A - e(\tilde{p})$; the dotted line exhibits the firm value difference in case of no frictions at all, amounting to $-\rho$. The exogenous parameter values used are: $A = 50$, $e(p) = 27p^2$, $\theta = 0.1$, $\eta = 2.5$, $\rho = U[0, \tilde{p}]$.

\textsuperscript{16}For such $\rho$-thresholds, the loan promise is at the level $b_N = b_{\text{max}}$ that maximizes expected loan value. The banks of firm-types with $\rho$ above those thresholds are not able to create enough collateral to raise the funds firms need. These firms are liquidated, obtaining zero value while their banks obtain the residual liquidation value after repayment of initial debt, $R - d_O$. 

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4 The social planner optimal allocations

We consider in this section a social planner (SP) that chooses which firms are continued and allocates output across agents in order to limit the amplification of lockdown losses described in Section 3. We show in the next section how the government can implement those optimal allocations in a decentralized manner.

4.1 The SP problem

We assume that the SP faces three constraints. First, she cannot observe firms’ effort, so allocations cannot be made effort contingent. Second, she cannot impose losses on the investors who hold banks’ old debt nor seize new investors’ funds at \( t = 0 \).\(^{17}\) Third, the SP cannot remove banks’ option to liquidate their customer firms and consume the residual part \( R - d_O \) of their recovery value at \( t = 1 \).

Formally, a SP allocation is described by a tuple

\[
\Lambda \equiv \left( \Gamma_C, (p(\rho))_{\rho \in \Gamma_C}, s, (c_{F,C}(z, \theta, \rho))_{\rho \in \Gamma_C}, (c_{F,L}(\theta, \rho))_{\rho \notin \Gamma_C}, c_B(\theta, \rho), c_I, \chi(\theta) \right),
\]

consisting of:

- The set of firm-types that continue, \( \Gamma_C \subset [0, \bar{\rho}] \). Firm-types in the set \( [0, \bar{\rho}] \setminus \Gamma_C \) are liquidated.
- For each firm-type \( \rho \in \Gamma_C \), an effort choice \( p(\rho) \in [0, \bar{\rho}] \).
- The funds provided by investors at \( t = 0 \), \( s \geq 0 \).
- For each firm-type \( \rho \in \Gamma_C \), the \( \theta \) and \( z \)-contingent firm consumption at \( t = 1 \), \( c_{F,C}(z, \theta, \rho) \geq 0 \) (where recall that \( z = A, 0 \) denotes the final payoff of a continued firm), and for each firm-type \( \rho \notin \Gamma_C \), the \( \theta \)-contingent firm consumption at \( t = 1 \), \( c_{F,L}(\theta, \rho) \geq 0 \).
- For each bank \( \rho \), the \( \theta \)-contingent bank consumption at \( t = 1 \) normalized by the measure \( h(\rho) \) of firms the bank finances, \( c_B(\theta, \rho) \geq 0 \).

\(^{17}\)In reality, authorities might be unwilling to impose losses to investors in banks’ debt because this could trigger a banking crisis with negative effects on the economy, and they could be unable to seize investors’ financial wealth to the extent that it is not held domestically or can be easily transferred abroad.
• The overall consumption at $t = 1$ by the old and new investors that have financed the economy, $c_I$.

• The $\theta$-contingent debt expansion of the government at $t = 1$, $x(\theta)$.

We say that a SP allocation $\Lambda$ is feasible if it satisfies the following constraints:

• Funds provided by investors at $t = 0$ are sufficient to finance the cash-flow shortfall of continuing firms:

$$s = \int_{\rho \in \Gamma_C} \rho h(\rho) d\rho. \quad (12)$$

• Aggregate $\theta$-contingent resource constraint at $t = 1$:

$$\int_{\rho \in \Gamma_C} \theta p(\rho) A h(\rho) d\rho + \int_{\rho \not\in \Gamma_C} R h(\rho) d\rho + x(\theta) = \int_{\rho \in \Gamma_C} E[c_{F,C}(z, \theta, \rho) | \theta, p(\rho)] h(\rho) d\rho + \int_{\rho \not\in \Gamma_C} c_{F,L}(\theta, \rho) h(\rho) d\rho + \int c_B(\theta, \rho) h(\rho) d\rho + c_I, \quad (13)$$

where the LHS accounts for overall resources available at $t = 1$ conditional on $\theta$ and takes into account that for $\rho \in \Gamma_C$, a fraction $\theta p(\rho)$ of the firms succeed, while for $\rho \not\in \Gamma_C$, liquidated firms yield the value $R$, and the government provides additional resources through its debt expansion $x(\theta)$. The RHS describes how available resources are used for consumption at $t = 1$ by firms, banks, and investors.

• Investors’ consumption at $t = 1$ is safe and compensates for the overall amount of old and new financing they have provided:

$$d_O + s \leq c_I. \quad (14)$$

• For $\rho \in \Gamma_C$, bank $\rho$ does not liquidate its firms to seize their residual recovery value:

$$E[c_B(\rho, \theta)] \geq R - d_O. \quad (15)$$

• For $\rho \not\in \Gamma_C$, bank $\rho$ consumes for all $\theta$ at least the residual recovery value of its firms:

$$c_B(\rho, \theta) \geq R - d_O. \quad (16)$$

• For $\rho \in \Gamma_C$, firms’ effort $p(\rho)$ coincides with that optimally chosen by the firms given their consumption allocation:

$$p(\rho) = \arg \max_{p'} \left\{ E \left[ \theta p' c_{F,C}(A, \theta, \rho) + (1 - \theta p') c_{F,C}(0, \theta, \rho) \right] - e(p') \right\}. \quad (17)$$
We denote with $\mathcal{F}$ the set of feasible allocations. For $\Lambda \in \mathcal{F}$, social welfare is given by

$$Y(\Lambda) \equiv E\left[\int_{\rho \in \Gamma_C} E[c_{F,C}(z,\theta,\rho)\theta,p(\rho)]h(\rho)d\rho + \int_{\rho \not\in \Gamma_C} c_{F,L}(\theta,\rho)h(\rho)d\rho\right] + E\left[\int_{\rho \not\in \Gamma_C} c_B(\theta,\rho)h(\rho)d\rho\right] + (c_I - d_O - s) + E\left[u_G(x(\theta))\right].$$

(18)

The four terms account for the utility of firms, that of banks, the excess utility of investors, and the disutility of the government.

Using resource constraints (12) and (13), and that $E[\theta] = 1$, we can more compactly rewrite the welfare expression in (18) and state the SP optimal allocation problem as:

$$\arg\max_{\Lambda \in \mathcal{F}} Y(\Lambda) = \int \left[1_{\rho \in \Gamma_C} (p(\rho)A - e(p(\rho)) - \rho) + 1_{\rho \not\in \Gamma_C} R\right] h(\rho)d\rho - E\left[g\left((x(\theta) - X)^+\right)\right].$$

(19)

The two terms in the social welfare expression above capture the overall firms’ value and the expected deadweights costs from a government debt expansion above its fiscal slack $X$, respectively.

### 4.2 Properties of optimal allocations

We present next a sequence of results that allows to characterize optimal SP allocations.

**Preliminary consumption properties.** We start providing some intuitive consumption properties an optimal allocation $\Lambda^*$ must satisfy. We denote from hereon optimal allocation variables with a * superscript.

Using Assumption 1, we have from the welfare expression in (19) that for $\rho \in \Gamma_C^*$ social welfare is strictly increasing in $p^*(\rho)$. In addition, if $p^*(\rho) < \overline{p}$, the effort optimality condition in (17) implies that:

$$e'(p^*(\rho)) = E\left[\theta (c^*_{F,C}(A,\theta,\rho) - c^*_{F,C}(0,\theta,\rho))\right].$$

(20)

The firm’s effort is thus increasing in the effort sensitivity of firm’s expected consumption upon success, $E\left[\theta c^*_{F,C}(A,\theta,\rho)\right]$, and decreasing in that upon failure, $E\left[\theta c^*_{F,C}(0,\theta,\rho)\right]$.

The SP will hence try to increase the consumption of firms that continue and succeed. From (19) and the aggregate resource constraint (13), we have that this can been achieved by minimizing the consumption of firms that continue and fail, firms that are liquidated, banks.
and investors, and using all the “costless” resources $X$ of the government.

The next result follows building on these intuitions.

**Lemma 2.** Optimal consumption allocations satisfy:

- **Firms:** consumption is zero if they continue and fail at $t = 1$ or if they are liquidated at $t = 0$.
- **Investors and banks:** their minimum consumption constraints in (14), (15) and (16) are binding.
- **Government:** its debt expansion at $t = 1$ strictly exceeds its fiscal slack $X$ for all $\theta$, $x^*(\theta) > X$, and its expected debt expansion, $\bar{x} \equiv E[x^*(\theta)]$, does not exceed the overall cash-flow shortfall of firms that continue, that is,

$$\bar{x}^* \leq \int_{\rho \in \Gamma_C^*} p \rho h(\rho) d\rho. \quad (21)$$

**Continuation of firms and their effort.** We next derive some properties of the optimal set of firms that continue and their effort. For convenience, we define the average project payoff of a firm-type $\rho \in \Gamma_C^*$ that is allocated to outsiders at $t = 1$ as:

$$b^*(\rho) = A - E[\theta c^*_{F,C}(A, \theta, \rho)]. \quad (22)$$

Using that $c^*_{F,C}(0, \theta, \rho) = 0$ (Lemma 2), we have from (2) and (17) that:

$$p^*(\rho) = \hat{p}(b^*(\rho)), \quad (23)$$

where the properties of the function $\hat{p}(\cdot)$ are exhibited in Lemma 1.

In addition, since (14), (15) and (16) are binding (Lemma 2), taking expectations in (13) we have the following value for continuation identity:

$$\int_{\rho \in \Gamma_C^*} \left( \underbrace{R - d_O + d_O}_{\text{Bank}} + \underbrace{\rho}_{\text{Old investors}} + \underbrace{\rho}_{\text{New investors}} \right) h(\rho) d\rho = \int_{\rho \in \Gamma_C^*} \hat{p}(b^*(\rho)) b^*(\rho) h(\rho) d\rho + \underbrace{\bar{x}^*_G}_{\text{Gov.}}. \quad (24)$$

The LHS in this expression can be interpreted as the value required for the continuation of the firms in the set $\Gamma_C^*$ by: their banks, the investors holding these banks’ old debt, and the new investors that finance firms’ cash-flow shortfalls $\rho$. The RHS in (24) instead captures the value that is allocated to these agents, and includes the contribution from the output of successful firms’ and the additional resources from the government’s debt expansion.
Two optimal allocation properties can be derived from the welfare expression in (19) and the value for continuation identity in (24). First, for a given measure of firms that continue, the SP should choose those with lower cash-flow shortfall. This is because less funds are required for the continuation of those firms, and thus more value can be appropriated by the firms, increasing their effort and welfare. Second, the value allocated to outsiders from firms that continue must be constant across them. The reason is that the convexity of the disutility associated with firms’ effort renders also convex the moral hazard output costs from the allocation of firm value to outsiders. The overall value allocated to outsiders from the firms that continue should hence be equally “carved out” from all those firms, resulting in equal effort across them.

The next result formalizes our discussion above.

**Lemma 3.** An optimal SP allocation satisfies:

- Only firms with an operating cost below a threshold $\rho^* > 0$ continue, that is, $\Gamma^*_C = [0, \rho^*]$.
- For all firms that continue, the average project payoff allocated to outsiders is a constant $b^*$ that only depends on $\rho^*$ and $\bar{\pi}^*$ and is determined by the equation:
  $$(R + E[\rho|\rho \leq \rho^*]) H(\rho^*) = \hat{p}(b^*)b^*H(\rho^*) + \bar{\pi}^*. \tag{25}$$
- The effort of all firms that continue is a constant $p^*$ satisfying $p^* = \hat{p}(b^*)$.

Note in particular that the properties in the lemma allow to write the value for continuation identity in (24) in the compact expression in (25). This expression illustrates the role of the government support in limiting the firm-bank linkages described in Section 3: For a given set of continuing firms $\Gamma^*_C$, as the government increases its expected debt $\bar{\pi}^*$ to finance support, the average firm payoff allocated to outsiders $b^*$ can be reduced, which in turn increases firms’ effort $\hat{p}(b^*)$ and, from (19), also welfare. Government support thus limits the amplification of lockdown losses, but entails deadweight costs once its fiscal slack has been exhausted. A trade-off ensues.

**Safe collateral, aggregate risk insurance and the government debt target.** Firms’ output is exposed to future aggregate shocks, but the investors that provide funds to meet firms’ cash-flow shortfalls during the lockdown require safe consumption. There is thus a need for aggregate risk insurance. We next analyze how it should be optimally provided.
Using Lemma 2 and 3, the aggregate resource constraint (13) at \( t = 1 \) can be written as:

\[
\left( d_O + E[p|\rho \leq \rho^*] \right) H(\rho^*) = \theta p^* A H(\rho^*) + \hat{x}^*(\theta) - \int_0^{\rho^*} \left( \theta p^* c_{F,C}^*(A, \theta, \rho) + c_B^*(\theta, \rho) \right) h(\rho) d\rho.
\]

(26)

The LHS in the expression above captures the payoffs required to compensate the investors that finance continuing firms, which should be repaid in all \( \theta \)-contingencies. It can be interpreted as investors’ demand for safe collateral. The RHS captures the \( \theta \)-contingent resources in the economy available for investors’ consumption. They include the overall resources from the firms’ output and the government’s debt expansion net of consumption by firms and banks.

Firms’ output decreases when \( \theta \) is low and expression (26) shows that investors’ demand for safe collateral can be accommodated in two ways. First, through a reduction in the consumption of successful firms, \( c_{F,C}^*(A, \theta, \rho) \), and/or banks, \( c_B^*(\theta, \rho) \). But, there is a limit to how much aggregate risk insurance these agents can provide due to their non-negative consumption constraints. Second, aggregate risk insurance can be provided through an increase in government debt, \( x^*(\theta) \). But, a \( \theta \)-contingent public debt level is costly due to the convexity of the deadweight cost function \( g(\cdot) \).

The next result on optimal provision of aggregate risk insurance can be proven.

**Lemma 4.** Let \( \Lambda^* \) be an optimal SP allocation and \( p^*, \rho^*, \bar{x}^* \) the associated allocation variables. There exists \( \hat{x}^* \leq \bar{x}^* \) such that the optimal \( \theta \)-contingent government debt level at \( t = 1 \) is given by

\[
x^*(\theta) = \max \{ \hat{x}^*, (d_O + E[p|\rho \leq \rho^*] - \theta p^* A) H(\rho^*) \}.
\]

(27)

In addition, \( \hat{x}^* \) is determined by the condition \( E[x^*(\theta)] = \bar{x}^* \). Finally:

- \textbf{If safe collateral is abundant given } \( \Lambda^* \), that is, if
  \[
  (d_O + E[p|\rho \leq \rho^*]) H(\rho^*) \leq \theta p^* A H(\rho^*) + \bar{x}^*,
  \]
  then \( \hat{x}^* = \bar{x}^* \), and \( x^*(\theta) = \bar{x}^* \) for all \( \theta \).

- \textbf{If safe collateral is scarce given } \( \Lambda^* \), that is, if (28) is not satisfied, then \( \hat{x}^* < \bar{x}^* \). In addition, there exists \( \theta^* > \theta \) such that for \( \rho \in \Gamma^*, \theta \in [\hat{\theta}, \theta^*] \) we have \( c_{F,C}^*(A, \theta, \rho) = c_B^*(\theta, \rho) = 0 \), \( x^*(\theta) > \hat{x}^* \). Finally, \( x^*(\theta) > \bar{x}^* \) for \( \theta \) close to \( \hat{\theta} \).

The lemma states that the optimal \( \theta \)-contingent \( t = 1 \) public debt level \( x^*(\theta) \) can be interpreted as arising from a constant “target” \( \hat{x}^* \) that is weakly below the expected level \( \bar{x}^* \). The
government meets this target for all aggregate shocks when safe collateral is abundant under the optimal policy, that is, when (28) is satisfied. In such a case, public debt is constant and all the aggregate risk insurance is provided by firms and banks. This entails no welfare cost, so that investors’ demand for safe consumption does not reduce aggregate welfare relative to that in an economy in which investors were risk-neutral and hence willing to purchase risky debt or equity from banks. Instead, when safe collateral is scarce under the optimal policy, the government cannot meet its debt level target $\hat{x}^*$ for bad aggregate shocks, that is, for $\theta < \theta^*$ (where the threshold $\theta^*$ introduced in the lemma satisfies $\theta^* > \theta$). In those contingencies, public debt is above target in order to provide safe consumption to investors.\footnote{In addition, the debt level exceeds its average $\bar{x}^*$ for shocks close to the worst one. Notice that these properties are consistent with the statement in Lemma 4 that, when there is safe collateral scarcity, the government’s debt target is strictly below its expected level, that is, $\hat{x}^* < \bar{x}^*$.} Importantly, to minimize the public debt volatility, whenever the government exceeds its debt target $\hat{x}^*$ the consumption of firms that continue (even if they succeed) and their banks is zero.

4.3 The fiscal slack and the optimal SP allocations

The properties of optimal allocations in Lemma 2, 3 and 4 allow to describe any possible optimal allocation by a tuple $(\rho^*, p^*, \bar{x}^*)$ consisting of a liquidation threshold $\rho^*$, an effort $p^*$ for the firms that continue, and an expected government debt $\bar{x}^*$. The tuple is feasible, which we denote as $(\rho^*, p^*, \bar{x}^*) \in \mathcal{F}$, if a solution $b^*$ to (25) exists and satisfies $p^* = \hat{p}(b^*)$. In that case, the government $\theta$–contingent optimal debt is described in Lemma 4, can be denoted with $x^*(\theta | \rho^*, p^*, \bar{x}^*)$ and is constant if and only if the safe collateral abundance condition given $(\rho^*, p^*, \bar{x}^*)$ in (28) is satisfied.

Given this notation, the SP optimal allocation problem in (19) can be written in a compact form as:

$$\arg \max_{(\rho^*, p^*, \bar{x}^*) \in \mathcal{F}} Y(\rho^*, p^*, \bar{x}^*) \equiv (p^* A - e(p^*) - E[\rho \leq \rho^*]) H(\rho^*) + R (1 - H(\rho^*))$$

$$- E \left[ g \left( (x^*(\theta | \rho^*, p^*, \bar{x}^*) - X)_{+} \right) \right].$$  (29)

Notice that both the solution to this problem, and whether or not it exhibits safe collateral abundance, will depend on the government’s fiscal slack $X$. The next formal result describes such dependence.
Proposition 1. Let \( X \) be the fiscal slack of the government. There exists a unique optimal allocation that can be described by a liquidation threshold, \( \rho^* (X) \), an effort of firms that continue, \( p^* (X) \), and an expected government debt, \( \bar{x}^* (X) \). We have that:

- \( \rho^* (X), p^* (X), \bar{x}^* (X) \) are increasing in \( X \), with \( d\bar{x}^* (X) / dX < 1 \).
- There is safe collateral abundance at the optimal allocation, that is, (28) is satisfied given \( \rho^* (X), p^* (X), \bar{x}^* (X) \) if and only if \( X \geq X_S \), where \( X_S \geq 0 \) and \( X_S > 0 \) if \( \theta \) and \( \eta \) are sufficiently small.

Proposition 1, which constitutes the main result of this section, describes the dependence of the solution to the SP problem on the government’s fiscal slack \( X \). The results are illustrated in Figure 2, where full lines in the panels represent optimal allocation variables under our baseline economy in which investors demand safety, and segmented lines represent those optimal allocations in an alternative benchmark economy in which investors were risk-neutral.

When the government has a low fiscal slack (\( X < X_S \)), the cost of expanding government debt is high, so the optimal expected government debt \( \bar{x}^* \) at \( t = 1 \) is endogenously low and a large fraction of the firms’ cash-flow shortfalls has to be financed by investors. The need of aggregate risk insurance to provide safe consumption to investors is consequently large and cannot be completely accommodated out of reductions in the consumption of successful firms and banks. Some aggregate risk insurance is thus optimally provided by the government, whose debt upon the worst shock exceeds its target (panel 2a)). Since public debt volatility is costly, the optimal policy design distorts the allocation of resources (relative to a risk-neutral environment): reducing the measure of firms that continue (panel 2b)) and increasing the support to each of them, which increases their effort (panel 2c)). The reason is that, when investors demand safety, a marginal increase in the effort of firms that continue expands the safe collateral value of their output (amounting to \( \theta p^* (X) A H (\rho^* (X)) \)), and reduces the need of costly aggregate risk insurance provision by the government (see (27)). Finally, the distortions in the optimal allocation implied by the requirement to provide safe consumption to investors reduce aggregate welfare relative to a risk-neutral economy (panel 2d)).

As the government’s fiscal slack increases, the larger access to costless resources to support the economy allows more firms to continue and they do so with larger effort. Expected public debt increases with the fiscal slack at a rate below one, so that the debt in excess of the fiscal...
slack and the associated deadweight costs, get reduced. All these effects lead to an increase in aggregate welfare. When fiscal slack is sufficiently large \( (X_S \leq X \leq E[\rho]) \), the expected government debt expansion \( \bar{x}^* \) is large and firms continue with high effort \( p^* \), both forces increase the amount of safe collateral in the economy (see (28)) so that safe collateral is not anymore scarce: the government achieves its debt target even upon the worst shocks, its debt level is constant, and all the aggregate risk insurance demanded by investors is provided by firms and banks. Further increases in the fiscal slack keep on increasing aggregate welfare until the slack exceeds the overall cash-flow shortfall of the corporate sector during the lockdown \( (X > E[\rho]) \), in which case the additional slack would only allow to make redistributive transfers with no aggregate welfare implications.

Summing up, the dependence of optimal support on fiscal slack in Proposition 1 shows that countries with low fiscal slack are less able to support firms, experience a larger drop in welfare and have more problems in stabilizing their public debt levels. These results are consistent with the empirical evidence in Augustin et al. [2022], which show that ex-ante fiscal capacity was an important amplifier of sovereign credit risk dynamics during the pandemic and that fiscally constrained countries were less resilient to the pandemic shock.

5 Implementation with government support policies

In this section, we focus on the decentralized implementation of SP optimal allocations through government support policies. We consider a government intervention package at the initial date consisting of three policies, each of them motivated by one of the three optimality lemmas in Section 4. First, some firm-specific transfers and taxes at \( t = 0 \) to achieve constant indebtedness and effort levels across all the firms that continue (motivated by Lemma 3). Second, the issuance of fairly priced government guarantees on bank debt to relax the bank’s funding constraints, eliminate the welfare destroying bank intermediation rents during the lockdown described in Section 3, and ensure bank profits remain at their minimum level (motivated by Lemma 2). Third, a procyclical taxation and subsidization of firms’ profits at \( t = 1 \) to pursue the government’s optimal debt target (motivated by Lemma 4).
Figure 2: Social Planner optimal allocation for given government fiscal slack

Note: The figure exhibits some allocation variables under the SP optimal solution in the baseline economy and in an alternative economy in which investors are risk-neutral, for different values of the fiscal slack $X$. The SP problem under risk-neutrality is given by (29) under the restriction $x^*(\theta|\rho^*, p^*, \bar{x}^*) = \bar{x}^*$ regardless of whether (28) is satisfied. Panel a): optimal government debt at $t = 1$ under $\theta = \bar{\theta}$ relative to its target, $x^*(\theta|\rho^*, p^*, \bar{x}^*) - \hat{x}^*$, where $x^*(\theta|\rho^*, p^*, \bar{x}^*)$ and $\hat{x}^*$ are defined in Lemma 4 for the baseline economy and $x^*(\theta|\rho^*, p^*, \bar{x}^*) - \hat{x}^* = 0$ for the risk-neutral economy. Panel b): optimal liquidation threshold, $\rho^*$. Panel c): optimal effort-choice of continuing firms $p^*$. Panel d): welfare relative to no lockdown: $Y(\rho^*, p^*, \bar{x}^*) - Y_0$, where $Y(\rho^*, p^*, \bar{x}^*)$ is defined in (29) and $Y_0 = \bar{p} A - c(\bar{p})$. The exogenous parameter values used in the numerical illustration are: $A = 50$, $c(p) = 27p^2$, $\bar{\theta} = 0.1$, $\eta = 2.5$, $\rho = U[0, \bar{\rho}]$ and $g(x) = (x - X)^2$. 
We next describe each of the policies in the government intervention. Let $\rho^*, p^*, \bar{x}^*$ denote for the rest of the section variables under the optimal SP allocation.

**Policy I: Firm-specific transfers and taxes at initial date.** The first government policy consists in type-specific oriented transfers to firms $T(\rho) \in \mathbb{R}$ at $t = 0$ to offset the heterogeneity in the magnitude of the lockdown shock across the firms that continue. Specifically, we define the cash-flow shortfall target

\[
\hat{\rho} \equiv E[\rho|\rho \leq \rho^*] - \bar{x}^*/H(\rho^*) \in [0, \rho^*),
\]

and the set of oriented government transfers defined by:

\[
T(\rho) = \begin{cases} 
\rho - \hat{\rho} & \text{for } \rho \leq \rho^* \\
0 & \text{for } \rho > \rho^* 
\end{cases}
\]

The transfer is negative for firms with small cash-flow shortfalls (types $\rho < \hat{\rho}$), which means that they are taxed at $t = 0$ and their funding needs increase as a result. The transfer is positive for firms with a medium cash-flow shortfall (types $\rho \in (\hat{\rho}, \rho^*]$), and it is zero for firms that are optimally liquidated (types $\rho > \rho^*$). The policy leads all firm-types $\rho \leq \rho^*$ to exhibit the same cash-flow shortfall of $\hat{\rho}$ at $t = 0$. We can henceforth consider that a single representative bank holds the legacy portfolio of all these firms and provides them new funding.

Note that, by construction, the overall funds the government needs at $t = 0$ to implement the set of transfers $T(\rho)$ amounts to

\[
\int_0^{\rho^*} T(\rho) h(\rho) d\rho = \bar{x}^*,
\]

that is, it coincides with the expected government debt level at $t = 1$ under the optimal allocation.\footnote{For internal consistency, and taking into account that in Section 2 we said that the government can issue debt $x$ at $t = 1$ to some unmodeled agents under some deadweight costs $g(x)$, one could assume that the government obtains the funds $\bar{x}^*$ at $t = 0$ by issuing safe debt to investors that is due at $t = 1$, and that at maturity of the initial public debt issuance it is reissued under the reduced form assumptions in Section 2.}

The remaining policies of the optimal policy package should hence be purely redistributive in expectation.

**Policy II: Fairly priced bank debt guarantees.** The second government policy is a fairly priced guarantee on the debt of the bank that finances firm-types $\rho \leq \rho^*$. No guarantee is instead issued on the debt of the banks that finance firm-types $\rho > \rho^*$.

Specifically, the fairly priced debt guarantee on the bank that finances firm-types $\rho \leq \rho^*$ is
described by a pair \((\kappa, \tau_B)\) consisting of: \(i)\) an aggregate shock threshold \(\kappa \in [\theta, 1]\), such that the government insures the repayment of debt at \(t = 1\) for aggregate shocks \(\theta \in [\theta, \kappa]\), and \(ii)\) an insurance fee \(\tau_B \geq 0\) the bank has to pay at \(t = 1\) for aggregate shocks \(\theta > \kappa\).\(^{20}\)

We say that a higher \(\kappa\) corresponds to a larger debt guarantee as it allows the bank to enjoy insurance for a larger set of aggregate shocks.

Given the guarantee \((\kappa, \tau_B)\), the lockdown bank financing to firm-types \(\rho \leq \rho^*\) consists of the provision of \(\tilde{\rho}\) units of funds at \(t = 0\) to each of these firms against a new overall promise \(b_N\) by the firms at \(t = 1\). As in the no support set-up in Section 2, we say that the new promise \(b_N\) for the lockdown financing is feasible if two conditions are satisfied:

- **Bank funding constraint:** the bank’s new and old debt is safe:
  \[d_O + \tilde{\rho} \leq \kappa \hat{p}(b_N)b_N,\]  \[(32)\]
  which differs from the analogous constraint in (4) in that the debt guarantee allows the bank to issue safe debt up to a fraction \(\kappa\) of the expected value of its loan portfolio. The guarantee provides aggregate risk insurance to the bank and relaxes the maximum leverage constraint imposed by investors’ demand for safe debt. This in turn allows the bank to set a lower loan promise \(b_N\) on the new financing to the firms.

- **Bank minimum profit constraint:** the bank prefers to provide new financing to the firms rather than to liquidate them:
  \[\Pi(b_N|\tau_B) \equiv E\left[(\theta \hat{p}(b_N) - d_O - \tilde{\rho} - \tau_B)^+\right] \geq R - d_O.\]  \[(33)\]
  which, relative to the analogous constraint (5), also includes the guarantee fee \(\tau_B\) the bank has to pay at \(t = 1\).

As in Section 2, we assume that whenever a feasible promise for the lockdown financing given \((\kappa, \tau_B)\) exists, the bank chooses the minimum one, which we denote as \(b_N'(\kappa, \tau_B)\) (or \(b_N'\) for short) and refer to as the competitive loan promise. Finally, the debt guarantee \((\kappa, \tau_B)\) is fairly priced at the competitive promise \(b_N'\) when the expected government disbursement associated with the guarantee is equal to the expected insurance fee paid by the bank, that is, when:

\[E\left[(d_O + \tilde{\rho} - \theta \hat{p}(b_N')b_N')^+\right] = E\left[\min\left\{(\theta \hat{p}(b_N') - d_O - \tilde{\rho})^+, \tau_B\right\}\right].\]  \[(34)\]

\(^{20}\)We assume the repayment of the insurance fee is junior to that of the bank’s debt, so that even for \(\theta > \kappa\) the bank could partially default on the repayment of the fee.
We next analyze the outcome of the competitive bank financing of firms’ funding needs \( \hat{\rho} \) given a guarantee \((\kappa, \tau_B)\). As in the no support case, the competitive promise \( b'_N \) makes at least one of the constraints (32) or (33) binding. If the bank funding constraint (32) is binding, increases in the size \( \kappa \) of the guarantee allow the competitive bank to pledge a larger fraction of the value of its loans as safe collateral to investors and this leads to a reduction in the competitive promise \( b'_N \).

In addition, if the guarantee \((\kappa, \tau_B)\) is fairly priced, the bank profits in the LHS of (33) can be rewritten as:

\[
\Pi(b'_N|\tau_B) = (d_O + \hat{\rho}) \frac{1 - \kappa}{\kappa}. \tag{35}
\]

The expression, which extends that for the no support case in (9), decomposes the bank profits as the product of its debt, \( d_O + \hat{\rho} \), and the per unit of debt lending rents, \((1 - \kappa)/\kappa\). We have also from (10) that, for a fairly priced guarantee \((\kappa, \tau_B)\) with \( \kappa \) close to \( \theta \), the bank’s profits satisfy \( \Pi(b'_N|\tau_B) > R - d_O \). That is, when the size of the guarantee is small, the aggregate risk insurance it provides is also small, and bank profits (equity) must be high in order to ensure a sufficient cushion against losses that makes debt safe. Hence, the intermediation rents obtained by the bank from the new lending \( \hat{\rho} \) make its minimum profit constraint (33) slack as in the no support case. As the size \( \kappa \) of the debt guarantee increases, the larger insurance against aggregate risk allows the competitive bank to set a lower promise \( b'_N \). This in turn reduces the bank’s intermediation rents until constraint (33) binds.\(^{21}\)

Building on these intuitions we can prove the following result.

**Lemma 5.** Suppose at \( t = 0 \) the set \( T(\rho) \) of transfers to firms described in (31) has been introduced. There exist \( \kappa \geq \theta \) and \( \tau'_B \geq 0 \) such that if the government also introduces at \( t = 0 \) a debt guarantee \((\kappa', \tau'_B)\) with \( \kappa' \geq \kappa \) on the bank that finances firm-types \( \rho \leq \rho^* \), then the debt guarantee \((\kappa', \tau'_B)\) is fairly priced and the optimality properties in Lemma 2 and 3 are satisfied. In particular:

- **firm-types** \( \rho \leq \rho^* \) obtain their residual financing needs and continue with optimal effort, that is,

\[
\hat{p}\left(b'_N(\kappa', \tau'_B)\right) = p^*,
\]

and the profits of their bank are minimum.

- **firm-types** \( \rho > \rho^* \) are not able to obtain financing from their banks and are liquidated.

\(^{21}\)Note that in this comparative statics argument with respect to the guarantee size \( \kappa \), the guarantee fee \( \tau_B \) also changes in order for the fairly priced condition at the competitive promise \( b'_N = b'_N(\kappa, \tau_B) \) in (34) to be satisfied.
• the expected government debt level at $t = 1$ equals $\bar{x}^\ast$.

The lemma states that the joint introduction at $t = 0$ of suitably chosen firm-specific transfers and sufficiently large fairly priced bank debt guarantees allows to achieve the optimal firms’ continuation and effort outcomes, and the optimal expected final date public debt level. Note in particular that the policies are directed only to the firms (and their banks) for which continuation is optimal, while the non-support policy to the other firms (and their banks) induces their liquidation. The results also highlight that the provision of aggregate risk insurance through debt guarantees eliminates the intermediation rents banks would otherwise obtain. The amplification of losses through firm-bank linkages described in Section 3 is thus forestalled.

**Policy III: the fiscal surplus target and firms’ procyclical taxation at $t = 1$.** We have from Lemma 4 that the $\theta$–contingent public debt $x^\ast(\theta)$ under an optimal allocation can be interpreted as arising from a constant debt target $\hat{x}^\ast$ at $t = 1$ with $\hat{x}^\ast \leq \bar{x}^\ast$ that is achieved under all aggregate shocks except the worst ones in case the allocation exhibits safe collateral scarcity. Recall that policy I leads to the initial issuance of public debt amounting to $\bar{x}^\ast$, and that upon bad shocks at $t = 1$ the government faces disbursements due to the bank debt guarantees in policy II. Thus, our third policy in the intervention package aims at stabilizing the public debt level at $\hat{x}^\ast$ at $t = 1$ through the implementation of a taxation and subsidization of firms’ profits at $t = 1$.

Specifically, consider a $\theta$–contingent proportional tax $\tau_F(\theta) \leq 1$ on the profits at $t = 1$ of firms that continue. We allow for $\tau_F(\theta) \leq 0$ for some values of $\theta$, which can be interpreted as subsidy to firms’ profits. We assume that the fiscal policy is revenue neutral, which happens if and only if $E[\theta \tau_F(\theta)] = 0$. In fact, for a firm that continues with effort $p$ and new loan promise $b_N$, the expected net amount of resources taxed by the government through the policy $\tau_F(\theta)$ is:

$$E[\theta p \tau_F(\theta)(A - b_N)] = E[\theta \tau_F(\theta)]p(A - b_N),$$

which is zero if and only if $E[\theta \tau_F(\theta)] = 0$.

In addition, the effort $p$ chosen by the firm in presence of the revenue neutral fiscal policy
\( \tau_F(\theta) \) is given by:

\[
p = \arg \max_{p' \in [0, p]} \left\{ E[\theta p' \left(1 - \tau_F(\theta)\right) (A - b_N)] - e(p') \right\}
\]

\[
= \arg \max_{p' \in [0, p]} \left\{ p' (A - b_N) - e(p') \right\} = \hat{p} \left( b_N \right),
\]

(37)

where in the second equality we have used \( E[\theta] = 1, E[\theta \tau_F(\theta)] = 0 \), and in the last equality the definition of \( \hat{p} \) in (2). The expression shows that a revenue neutral fiscal policy at \( t = 1 \) is also effort neutral at \( t = 0 \).

We next informally describe how the revenue neutral policy \( \tau_F(\theta) \) can be constructed. The set of transfers to firms at \( t = 0 \) in policy I implies an initial public debt issuance of \( \bar{x}^* \). To achieve its final date debt target of \( \hat{x}^* \leq \bar{x}^* \) at \( t = 1 \) in all shocks \( \theta \), the government pursues a fiscal surplus of \( x^* - \hat{x}^* \geq 0 \) at \( t = 1 \) in all shocks \( \theta \). Given the fairly priced bank debt guarantee \((\kappa', \tau'_B)\) in policy II, the government achieves its target if the policy \( \tau_F(\theta) \) satisfies:

\[
\bar{x}^* - \hat{x}^* = \left[ \min \left\{ \tau'_B, \theta p^* b'_N (\kappa', \tau'_B) - d_O - \bar{\rho} \right\} + \theta p^* \tau_F(\theta) (A - b'_N (\kappa', \tau'_B)) \right] H(\rho^*),
\]

(38)

where the two terms in the RHS in the expression above capture the cash-flows received by the government due to the fairly priced debt guarantee and the firms’ profit taxation, respectively. Note that a positive (negative) value for these terms means the government exhibits a cash inflow (outflow).\(^{22}\)

For negative aggregate shocks (low \( \theta \)), the first term in the RHS of (38) is negative as the government has to satisfy the debt guarantee. In order to achieve the fiscal surplus target \( \bar{x}^* - \hat{x}^* \), the second term in the RHS of (38) has to be positive, that is, the government has to tax firms’ profits, \( \tau_F(\theta) > 0 \). In addition, the lower \( \theta \) the larger the taxation \( \tau_F(\theta) \) has to be because the debt guarantee disbursements increase and the measure of firms whose profits can be taxed decreases. Since the taxation capacity of the government is limited by the constraint \( \tau_F(\theta) \leq 1 \), it might be unfeasible to achieve the fiscal surplus target for sufficiently bad aggregate shocks. In those cases, which only arise when there is safe collateral scarcity, the firms’ taxation is \( \tau_F(\theta) = 1 \) to minimize the gap between the actual and targeted fiscal surplus levels.

\(^{22}\)In the expression in (38) we have assumed that, given the entire intervention package \((T(\rho), (\kappa', \tau'_B), \tau_F(\theta)))\), the endogenous firms’ effort is \( p' \). We have from (37) that this will be the case if the policy \( \tau_F(\theta) \) defined in (38) is revenue neutral, which is the case as we show in the proof of Proposition 2.
In contrast, for good aggregate shocks (high \( \theta \)), the first term in the RHS of (38) is positive as the bank is able to repay the guarantee fee \( \tau'_B \). It is possible to prove that the overall guarantee fee exceeds the fiscal surplus target (\( \tau'_B H(\rho^*) > \bar{x}^* - \hat{x}^* \)), so that the second term in the RHS of (38) is negative, that is, the government subsidizes firms’ profits, \( \tau_F(\theta) < 0 \).

The next result concludes our presentation of an optimal support toolkit.

**Proposition 2.** Let \( \rho^*, p^*, b^*, \bar{x}^*, \hat{x}^* \) denote variables associated with the optimal allocation. The government can induce the optimal allocation through the combination of:

I. The set of transfers and taxes to firms at \( t = 0 \) described by \( T(\rho) \) in (31).

II. A fairly priced guarantee (\( \kappa', \tau'_B \)) on the bank that finances firm-types \( \rho \leq \rho^* \) as described in Lemma 5.

III. A revenue neutral procyclical fiscal policy \( \tau_F(\theta) \) that results from the introduction of a fiscal surplus target of \( \bar{x}^* - \hat{x}^* \geq 0 \) at \( t = 1 \) that the government aims at achieving upon taxation and subsidization of firms’ profits at \( t = 1 \). Moreover, \( \tau_F(\theta) \) is given by:

\[
\tau_F(\theta) = \min \left\{ \frac{(\bar{x}^* - \hat{x}^*)}{H(\rho^*)} - \min \left\{ \frac{\tau'_B, \theta p^* b^* - d_O - \tilde{\rho}}{\theta p^* (A - b^*)} \right\}, 1 \right\}.
\] (39)

The proposition shows that the policy toolkit consisting of initial date transfers and taxes to firms (*policy I*), fairly priced bank debt guarantees (*policy II*) and future procyclical taxation of firms’ profits (*policy III*) is able to achieve optimality. The procyclical nature of the firms’ fiscal policy means that their profits will be taxed upon future bad shocks to (at least partially) finance the guarantees on bank debt issued during the lockdown, and those profits will instead be subsidized upon future good shocks using the proceeds from the insurance fees paid by the bank. Thus, our optimal policy highlights that optimally responding to the current crisis created by Covid-19 creates some future “fiscal policy legacies” that impose a procyclical post-lockdown taxation and may limit the government ability to provide support in future recessions. This is the aggregate welfare maximizing way to provide aggregate risk insurance while avoiding the costs associated with public debt sustainability problems.

Figure 3 illustrates the importance of the provision of aggregate risk insurance through policies II and III by comparing the outcome under the optimal toolkit consisting of policies I, II, and III (full line) with that under the sole use of the transfers to firms in policy I (segmented
line) for different values of the government’s fiscal slack $X$. The difference between segmented and full lines in the panels in the figure can thus be interpreted as the incremental effect of the provision of aggregate risk insurance once firm-specific transfers have been granted. First, the figure highlights that, for low values of $X$, transfers are not enough for the continuation of firms when only policy $I$ is used, while firms are able to continue when aggregate risk insurance is provided through policies $II$ and $III$. For larger values of $X$, the two policy toolkits allow the firms’ continuation but the competitive promise $b'_N$ for the residual financing $\hat{\rho}$ demanded by firm-types $\rho \leq \rho^*$ is lower when aggregate risk insurance is provided (Panel 3a). This is because the debt guarantees in policy $II$ allow the bank to increase its leverage, that is, to pledge a larger fraction of the value of its loan portfolio as safe collateral to issue debt (Panel 3b). When only transfers are used, the bank must provide all the aggregate risk insurance to back the issuance of new safe debt, and in equilibrium its equity must increase, leading to higher profits than in the no lockdown economy. Instead, when the bank’s debt guarantee is as high as dictated by policy $II$, bank profits coincide with its minimum optimal level (Panel 3c). Finally, the guarantees in policy $II$ lead to government disbursements upon bad shocks at the final date, and the optimal intervention requires the taxation of firms in policy $III$ and some additional government debt expansion when its fiscal slack is small, in which case the debt level target is missed (Panel 3d).²³

Note that as the government’s fiscal slack increases, the differences between the outcomes under the two policy interventions get narrowed. The reason is that larger fiscal slack allows the government to directly cover more of the cash-flow shortfall of the firms that continue. This reduces the amount of debt raised by the bank, and the incremental effect of the aggregate risk insurance provided by the government through policies $II$ and $III$ also diminishes.

²³Panel 3d shows that for $X < X_S$, that is, when there is safe collateral scarcity, the overall taxation of firms upon the worst shock is increasing in $X$. This is because in that region the government misses its debt target and taxes entirely firms profits. Overall firm profits in turn increase with $X$ because each firm that continues is less indebted and has a higher probability of success, and more firms continue.
Figure 3: Equilibrium under optimal policies and only transfers for given fiscal slack.

Note: The figure exhibits some equilibrium variables under the optimal policy toolkit comprising policy I, II and III (solid line) and under the sole use of policy I (segmented line), for different values of the fiscal slack $X$. Under the two policy toolkits, the use of policy I is as described in $T(\rho)$ in (31). Panel a): competitive new loan promise, $b^*_N$. Panel b): bank’s leverage, $\frac{d_O + \hat{\rho}}{\hat{\rho}(b^*_N)}$. Panel c): bank’s profits, $\hat{\rho}(b^*_N) b^*_N - d_O - \hat{\rho}$. Panel d): overall firms’ taxation $\min \{ \tau_F(\theta) \hat{\rho}(b^*_N) (A - b^*_N) , d_O + \hat{\rho} - \theta p^* b^*_N \} H(\rho^*)$ and debt expansion $x^*(\theta) - x^*$ at $t = 1$ under the worst shock $\theta = \underline{\theta}$ (dotted line), in optimal policy toolkit. The exogenous parameter values used are: $A = 50$, $e(p) = 27p^2$, $\theta = 0.1$, $\eta = 2.5$, $\rho = U[0, \bar{\rho}]$ and $g(x) = (x - X)^2$. 

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6 Policy discussion

We have so far described the properties of optimal support to firms in a lockdown and presented a possible implementation. We devote this section to discuss through the lens of our results different aspects of the multi-front set of policies that fiscal, supervisory and monetary authorities across the globe have introduced to help firms withstand the economic fallout of Covid-19 lockdowns.

6.1 Support policies to small and medium enterprises

Our model focuses on bank dependent firms, which typically correspond with small and medium enterprises (SMEs). We next discuss how the actually observed support policies to these firms compares to the optimal policy toolkit described in the previous section.

Use of transfers. Different types of transfers to firms have constituted a cornerstone of support policies in many jurisdictions. In the US, transfers in order to cover payroll costs and other operating costs of firms whose activity had been affected by Covid-19 were introduced in March 2020 through the forgivable loans in the Paycheck Protection Program. In most European countries, transfers in the form of subsidies to the payment of wages of furloughed workers were introduced in March 2020 with the objective to protect the long-term value of employer-employee relationships. These policies targeted some dimensions of the heterogeneity in liquidity needs by firms, since typically these transfers were proportional to payroll costs. Also, in a subsequent stage of the crisis and amid growing concerns on firms’ potential debt overhang problems, some governments introduced direct grants whose amount was proportional to some measure of the economic impact of lockdowns on the firms’ activity. For example, in Italy these grants to small businesses amounted to a fraction of the fall in revenues during the first lockdown period relative to the same period in 2019, and additional similar grants were introduced on the most hit sectors during subsequent lockdowns in Winter 2020. The policy included a minimum revenue drop threshold of 33% as eligibility criteria, so that mildly affected firms could not opt in, and a cap on the maximum amount of grants firms could receive, which limited the support to firms with largest revenue drops.24 Consistent with the policy recommendation of our model, the design of these government transfers was

24For details on the grants to firms introduced by the Italian Government see: here and here.
indexed to the magnitude of the impact of the Covid-19 crisis while including features that focused support on mediumly hit firms, thus avoiding the “waste” of too many resources on both mildly and too severely hit firms.

**Use of guarantees.** A second central element of the support to SMEs during lockdowns consisted of several policies aimed at enhancing firms’ access to bank funding that share the features of a public guarantee. This was explicitly the case of the government guarantees on the new loans to firms issued by banks which were introduced by most countries after the Covid-19 outbreak (see IMF [2021b]). These schemes were not intended as a form of short-term bridge financing, but rather as a source of medium-or-long term funding to the firms.\(^{25}\) These loan guarantees expose fiscal authorities to significant firm default risk, and would lead to important public disbursements in the future if negative shocks that lead to the failure of many firms materialize. The policy thus amounts to a transfer of aggregate loss exposure from banks to the public, and, similarly to guarantees on banks’ debt, would also relax the funding constraint of a bank that needs safe collateral to raise funding.\(^{26}\)

Also, supervisory authorities in many jurisdictions contributed to support bank lending during the pandemic by releasing macroprudential capital buffers like the Countercyclical Capital buffer (CCyB) or through forbearance measures in the prudential treatment of risky exposures.\(^{27}\) These measures were intended to allow banks operate with a higher leverage, which is also the intended effect of introducing the bank debt guarantees that we emphasize as part of the optimal intervention toolkit in Section 5. Relatedly, it is possible to prove that the introduction of guarantees on the debt of the bank in our model, which is funded by investors that demand safe collateral, would be isomorphic to the relaxation of the capital requirements of a bank that has explicit access to the safety net through its funding with insured deposits. So, consistent with the implications of our model, the actions of supervisory authorities to support bank lending to firms could be interpreted as a way to further public

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\(^{25}\)As an example, the average maturity of loans with guarantees issued during the Covid-19 crisis in Euro area countries was five years (ECB Economic Bulletin, Issue 6/2020).

\(^{26}\)See Section 6.2 for a further discussion of the differences between the two types of guarantees.

\(^{27}\)For example: The Bank of England swiftly reacted to the March 2020 lockdowns by reducing the CCyB from 1% to 0% (see here); the European Banking Authority recommended in March 2020 supervisory authorities to make full use of the flexibility in the regulatory framework to ensure that “prudential regulation is countercyclical and banks can provide the necessary support to the household and corporate sectors” (see here). The Fed reduced bank reserve requirement ratios to 0% in March 2020, and communicated that the action would “help to support lending to households and businesses.” (see here).
insurance against aggregate risk.\textsuperscript{28}

Our model has also implications on the optimal relative use of transfers versus guarantees. Governments with lower fiscal slack spend less in transfers (see Proposition 1 and 2), which then makes the provision of aggregate risk insurance through guarantees more necessary (see Figure 3b, and recall that the larger the debt guarantees the higher the leverage the bank can take). Using the cross-country data on support policies in Gourinchas et al. [2021], Figure 4 presents a scattered plot with the ratio of the announced size of guarantee–like policies versus the overall announced size of guarantee–like cum transfer-like policies (vertical axis) and the ratio of public debt to GDP (horizontal axis), which can be interpreted as an inverse measure of fiscal slack. The figure points towards some positive (albeit weak) correlation across the two measures, so that a larger relative use of guarantees is associated with fiscally weaker countries. These results would be consistent with our model implications. Yet, the weak correlation suggests many countries might have taken a policy mix that, through the lens of our model, is far from optimal. In this respect, the evidence in Jappelli et al. [2021] that in the European Union the contingent government liabilities associated with interventions led to an increase in the elasticity of firms to sovereigns’ CDS in countries with larger fiscal capacity, but not in those with lower fiscal capacity, also suggests that counties might have supported firms with a suboptimal mix of guarantees versus transfers.

\textbf{Lack of fiscal consolidation plans.} The interventions during the pandemic have led to an increase in fiscal deficits and government debt levels, which have reached their highest levels over the past several decades and could increase further if the contingent liabilities associated with public guarantees materialize. Yet, government interventions have not come along with credible future fiscal consolidation plans, which could be one of the factors driving the current more persistent than expected inflationary pressures in many economies.\textsuperscript{29}

The missing fiscal consolidation programme is, in our view, the main suboptimality dimension in the economic policies in the Covid-19 crisis. Recall that our results in Section 5

\textsuperscript{28}The capital releases or supervisory forbearance measures were not accompanied by an “insurance fee”, so they would be formally interpreted within our model as non-priced bank debt guarantees.

\textsuperscript{29}For instance, Cochrane [2022] argues that the Covid interventions in the US came without “a corresponding increase in expectations that the government would, someday, raise surpluses by $5 trillion in present value to repay the debt” and did not include “motions about long-run fiscal planning, long-run deficit reduction, and entitlement and tax reform in 2020-2021.”
emphasize the need of a procyclical future fiscal policy which aims at reducing (or stabilizing) public debt given the initial debt expansion and the explicit liabilities incurred during the crisis. International economic institutions have recently emphasized the need for fiscal policy reform aiming to debt reduction after the pandemic, and have set guidance on effective post-crisis revenue generation for tax administrations (see IMF [2021a], IMF [2021c]). However, as noted by Kose et al. [2021], pursuing debt reduction policies through tax and fiscal surplus reforms can face difficult technical, practical and political obstacles.

6.2 Lack of future fiscal policy and the ranking of guarantees

In this section, we further analyze the welfare implications of a government’s inability to set its future fiscal policy. We consider in this environment a government that at $t = 0$ can make transfers to firms and issue one out of two types guarantees: on bank loans, as many governments did during the crisis, or on bank debt and fairly priced, which instead has not been part of the observed policy response. Our analysis shows that the second type of guarantee is preferable as a tool to provide aggregate risk insurance when there are limits on the future state contingent fiscal policy path.

Consider a constrained support toolkit consisting of a set of firm-specific transfers $T(\rho)$ as
defined in (31), and a guarantee on the new bank lending to firms described by the fraction \( \gamma \in [0, 1] \) of the additional promise \( b_N - b_O \) that the government pays to the bank in case of a firm’s failure.\(^{30}\)

Analogously to Section 5, the set of transfers \( T(\rho) \) partially eliminate heterogeneity across firms, and allow to focus on the continuation of firm-types \( \rho \) below some threshold \( \rho^* \) and with a constant cash-flow shortfall \( \bar{\rho} \).

Given the loan guarantee \( \gamma \), the funding constraint of the bank that finances these firms takes the form:

\[
d_O + \bar{\rho} \leq \theta \hat{p}(b_N) b_N + (1 - \theta \hat{p}(b_N)) \gamma (b_N - b_O),
\]

in which the new term in the RHS relative to (4) captures that under the shock \( \theta \) a fraction \( 1 - \theta \hat{p}(b_N) \) of the firms default and the bank obtains the government guaranteed amount \( \gamma (b_N - b_O) \).

Thus, similarly to guarantees on bank debt, guarantees on bank loans provide aggregate risk insurance, relax the bank funding constraint, and support cheaper financing to firms.

But, are the two types of guarantees an equivalent way to reduce the funding cost of firms? The answer is no. To see this, consider a loan guarantee \( \gamma \) and a fairly priced bank debt guarantee \((\kappa, \tau_B)\) that induce an identical competitive new promise \( b_N' \) in which the bank funding constraints with loan guarantees in (40) and with debt guarantees in (32) are both binding, that is:

\[
d_O + \bar{\rho} = \theta \hat{p}(b_N') b_N' + (1 - \theta \hat{p}(b_N')) \gamma (b_N' - b_O) = \kappa \hat{p}(b_N') b_N'.
\]

The two guarantees thus lead to the same overall firm value (amounting to \( \hat{p}(b_N') A \) for each of the firms that continue) and government overall disbursement upon the worst shock \( \theta \) at \( t = 1 \). They differ though on the government disbursements upon other shocks \( \theta \) by the following amount:

\[
\Delta(\theta) \equiv \left(1 - \theta \hat{p}(b_N') \right) \gamma (b_N' - b_O) H(\rho^*) - \max \left\{ d_O + \bar{\rho} - \theta \hat{p}(b_N') b_N', -\tau_B \right\} H(\rho^*),
\]

\(^{30}\)We do not include an insurance charge on firms nor on banks for access to the guarantee on loans because such feature was typically not included in the goverment support policies during the pandemic.
and it is easy to check using (41) that $\Delta(\theta) > 0$ for all $\theta > \hat{\theta}$.\footnote{For $\theta \in (\hat{\theta}, \kappa)$, we have from (41) and (42) that: $\Delta(\theta) = \Delta(\theta) - \Delta(\theta) = \left(\theta - \hat{\theta}\right) \hat{\rho} \left(b'_N\right) \left(b'_N - \gamma \left(b'_N - b_O\right)\right) H(\rho^*) > 0.$}

We have thus that, for a given safe collateral creation through guarantees, the government will have to make strictly larger disbursements in all shocks but the worst one if it uses loan guarantees than if it uses fairly priced debt guarantees. The reason is that debt guarantees directly target the insurance against aggregate risk of investors, while bank loan guarantees also provide some insurance against idiosyncratic firm risk, increasing disbursements. Those additional disbursements upon loan guarantees would be appropriated by the banks, which would make larger profits. The value redistribution from the government to the banks would be welfare reducing due to the deadweight costs from government support in excess of its fiscal slack. So, when the government cannot set a $\theta$–contingent fiscal policy in the future, it should rely on fairly priced bank debt guarantees rather than bank loan guarantees in order to provide aggregate risk insurance in the economy.\footnote{The arguments in this paragraph and (42) would also hold if the bank debt guarantee were not reimbursed, so the fair reimbursement of the guarantee is not a necessary feature for our findings in this section.}

### Comparison of the support toolkits

Figure 5 compares, for different levels of the government fiscal slack, the outcome under the optimal use of three intervention toolkits: \(i)\) firm-specific initial transfers and taxes, fairly priced bank debt guarantees and procyclical fiscal policy described by tuples $(T(\rho), (\kappa, \tau_B), \tau_F(\theta))$, which from Section 5 allows to achieve optimal allocations, in the orange solid-line (toolkit \(I)\), \(ii)\) firm-specific initial transfers and taxes and fairly priced bank debt guarantees $(T(\rho), (\kappa, \tau_B))$, in the green segmented-line (toolkit \(II)\), and \(iii)\) firm-specific initial transfers and taxes and bank loan guarantees $(T(\rho), \gamma)$, in the purple dotted-line (toolkit \(III)\).

Panel 5a shows the aggregate risk insurance provided by each policy under the optimal use of the toolkit, which can be defined as the difference between the overall safe collateral required by investors and the minimum return of bank loans, that is:

$$
\frac{\left(d_O + \hat{\rho}\right)H(\rho^*)}{\text{Required safe collateral}} - \frac{\hat{\theta}\hat{\rho}(b'_N)b'_N H(\rho^*)}{\text{Safe payoff bank loans}},
$$

where (43)
where $\rho^*, \tilde{\rho}, b'_N$ denote the optimal liquidation threshold, cash-flow shortfall upon continuation, and induced competitive new promise, respectively, under each of the toolkits.\footnote{\textsuperscript{33}In absence of policy intervention, the bank funding constraint (4) implies that expression (43) is non-positive.} Toolkit $I$ allows to optimally provide aggregate risk insurance through a $\theta$–contingent fiscal policy at $t = 1$ that minimizes government debt volatility (Proposition 2). In contrast, the lack of a fiscal policy at $t = 1$ under toolkits $II$ and $III$ implies that the provision of aggregate risk insurance through guarantees involves costly government debt volatility. As a result, the provision of aggregate risk insurance under the optimal use of these toolkits gets reduced, and the more so for toolkit $III$ as the use of loan guarantees to create safe collateral is more costly than that of bank debt guarantees (see (41)).

The constraints on the provision of aggregate insurance under the toolkits $II$ and $III$ have real effects. Panel 5b shows that a lower measure of firms are able to continue (higher liquidation threshold) in absence of a future fiscal policy. Panel 5c shows instead an increase in the effort that is optimally induced under toolkits $II$ and $III$ relative to that under toolkit $I$. This is because the increase in effort expands the safe collateral created by banks from firm loans (amounting to $\theta \tilde{\rho}(b'_N)b_N$), and the distortion along this margin partially reduces the need of aggregate risk insurance, which these suboptimal toolkits find costly to provide. Finally, Panel 5d shows that a lower firm continuation, a distortedly high firms’ effort and a more volatile government debt lead to reductions in welfare under the toolkits $II$ and $III$ relative to toolkit $I$. All these effects are stronger when loan guarantees are used instead of fairly priced bank debt guarantees, as they are a more costly way to provide aggregate risk insurance, and when the government has a low fiscal slack, as in this case there is a higher need of aggregate risk insurance provision.

6.3 Interventions in debt markets

We have thus far considered bank funded firms and analyzed the optimality of providing support indirectly through banks. We adapt in this section our framework to study support to firms that raise funding through debt markets. We then discuss some of the debt market interventions by monetary and fiscal authorities during the Covid-19 crisis.

Consider a variation of our model in which the firms’ payoff at $t = 1$ in case of success is $A + S$, and in case of failure is $S$, where $S > 0$. The strictly positive safe payoff $S$ allows firms
Figure 5: Comparison of intervention toolkits for given fiscal slack.

Note: The figure exhibits some allocation variables under the optimal use of different policy toolkits, for different values of the fiscal slack $X$. Toolkit I consists of the optimal policy mix $(T(\rho), (\tau_B, \tau_F(\theta)))$. Toolkit II consists of initial transfers and fairly priced bank debt guarantees $(T(\rho), (\kappa, \tau_B))$. Toolkit III consists of initial transfers and loan guarantees $(T(\rho), \gamma)$. Panel a): aggregate risk insurance provision, $(d_O + \hat{\rho})H(\rho^*) - \hat{\theta} p(b_N'(b_N') H(\rho^*))$. Panel b): firms' liquidation threshold, $\rho^*$. Panel c): effort of continuing firms, $p^*$. Panel d): welfare difference relative to no lockdown: $Y(\rho^*, p^*, x^*(\theta)) - Y_0$, where $x^*(\theta)$ for toolkit I is defined in Lemma 4 and for the other toolkits results from the cash-flows implied by the guarantees, $Y(\rho^*, p^*, x^*(\theta))$ is defined in (29) given $x^*(\theta)$, and $Y_0 = \bar{p} A - e(p)$. The exogenous parameter values used are: $A = 50, e(p) = 27p^2, \bar{\theta} = 0.1, \eta = 7.5, \rho = U[0, \rho]$ and $g(x) = (x - X)^2$. 

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to obtain debt financing directly from investors without the need of the intermediation of a bank that takes advantage of the diversification opportunities in the economy. For simplicity, we assume that firms are only financed through debt markets and that their \( t = 0 \) debt \( b_O \) is the maximum possible, that is, \( b_O = S \). The lockdown creates cash-flow shortfalls to firms and, since they had exhausted their capability to directly issue (safe) debt to investors, firms lose their access to debt markets. To avoid a credit crunch that leads to firms’ liquidation, yielding a recovery value that we assume equal to the safe payoff \( S \), a government can provide support to the economy. We allow in particular the government to set-up at \( t = 0 \) a vehicle that purchases debt from many issuer firms and that is financed through safe debt placed to investors. The rest of the model is left unmodified.

The government vehicle plays a role analogous to that of the bank in our baseline model. It is in fact easy to prove that, for the purposes of optimal policy design, this model is isomorphic to the baseline model with zero initial bank size and firms’ recovery value, that is, with \( b'_O = d'_O = R' = 0 \). Optimal SP allocations would be described by the sequence of lemmas in Section 4 with the only difference that there is no need to grant any consumption to some initial vehicle owners as opposed to the minimum consumption bank owners enjoy in the baseline model due to their option to liquidate firms. Optimal SP allocations could be implemented through the three policies described in Proposition 2, where in this variation of the model the guarantees would now be granted on the vehicle’s debt. In addition, since there is no initial vehicle owner to which some consumption must be granted, the fee the vehicle would pay to the government would in equilibrium amount to the entire residual claim of the vehicle, which effectively allows to interpret the government as the owner of the equity of the vehicle. So, the government sets up a vehicle whose debt is guaranteed up to some leverage, which leads to some disbursements upon bad shocks, and owns the equity of the vehicle, which generates profits upon good shocks. Finally, the provision of aggregate risk insurance in the debt market through a government sponsored vehicle would be supplemented with a procyclical taxation and subsidization of future firms’ profits to achieve a fiscal surplus target. In particular, the

\[ R' - d'_O = 0. \]

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\[ 34 \] Notice that Assumptions 2 - 5 can be satisfied with \( b'_O = d'_O = R' = 0 \) and \( \eta = 0 \).

\[ 35 \] Formally, under the equivalence between the debt funding model with a government sponsored vehicle and the baseline bank funding model with an initial bank size of zero, all the analysis in Section 4 would hold taking into account that the initial bank owners’ minimum consumption entering in the RHS of (15) and (16) is simply \( R' - d'_O = 0 \).
government would use any profits it obtains from the vehicle in excess of its fiscal surplus target to subsidize firms’ profits in the future.

The type of vehicle our model advocates for resembles the structure of the Commercial Paper Funding Facility and Primary Market Funding Facility set-up by the US Treasury and the Fed in March 2020. These programs set up special purpose vehicles to purchase newly issued commercial paper and debt of non-financial corporates affected by the Covid-19 outbreak and whose pre-crisis rating was investment grade. The Fed provided debt funding to the vehicles and the US Treasury injected sufficient equity to ensure the safety of the Fed’s investments. Notice that both in the actual vehicles set-up by the US Treasury and the Fed and in the one we have proposed, the fiscal authority provides aggregate risk insurance to back the creation of safe collateral demanded by investors (which in the actual implementation was the central bank). The difference in the two schemes is that equity injections amount to a pre-funding of the required aggregate risk insurance, while debt guarantees imply a state-contingent funding of such insurance. It is possible to prove that the two schemes would be equivalent provided they are accompanied with suitably designed procyclical fiscal policy. The results in this section thus help rationalize some of the observed combined interventions by governments and central banks and emphasize the necessary supplemental role of fiscal policy.

7 Conclusion

We analyze optimal government support to firms facing heterogenous revenue losses and financing needs during a lockdown. The analysis is conducted in a competitive model of intermediation that features loss amplification through the balance sheet linkages between firms and banks created by the interplay of two frictions. First, the increase in debt required by firms to survive the lockdown reduces their equity, creating moral hazard problems that reduce firms’ output and lead to losses on banks’ pre-lockdown loans. Second, banks must finance their lending to firms from investors which demand safety. The presence of both idiosyncratic and aggregate risk gives a role to banks’ equity in addition to firms’ equity. The need for aggregate risk absorption capacity to intermediate funds restricts bank lending

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36 See official term sheets for the two programs [here](#) and [here](#).
37 During the GFC, the US Treasury did not provide such type of first-loss absorption, which led the Fed to refrain from introducing facilities targeted to risky debt markets because that would have gone beyond its mandate (see Bordo and Duca [2020]).
supply and leads to bank intermediation rents, making firm borrowing more expensive and further amplifying output losses.

We consider a government that faces convex deadweight cost from debt issuance and show that its optimal support to firms combines three policies. First, transfers to firms contingent on their liquidity needs, which help repairing their net worth and alleviate the moral hazard induced by excessive indebtedness. Second, fairly priced guarantees on banks’ debt, which provide aggregate risk insurance to safety seeker investors and constitute a substitute for the loss absorption role of bank equity. These guarantees eliminate bank intermediation rents and indirectly repair firm net worth by reducing their borrowing costs.

While transfers imply an immediate increase in public debt, guarantees imply additional contingent liabilities for the government that could potentially increase public debt in the future. Therefore, these tools must come together with a fiscal consolidation program for the post-pandemic times in order to stabilize public debt. In particular, the third policy of the optimal support consists in a pro-cyclical future taxation of firms’ profits that aims at achieving a fiscal surplus target. Firms’ taxation must be higher upon bad shocks in the future to finance the payments implied by the guarantees issued during the lockdown. Our optimal policy thus highlights that optimally responding to the pandemic crisis creates future fiscal policy legacies that will limit the capability to provide support in future recessions. The stylized static framework developed in this paper could be embedded into a dynamic model and shed new insights on the effects of these legacies and on the policy intervention during recessions and their exit strategies. We leave these issues for future research.
References


Nicolas Crouzet and Fabrice Tourre. Can the cure kill the patient? corporate credit interventions and debt overhang. *Corporate credit interventions and debt overhang (June 1, 2021)*, 2021.


A Appendix

A.1 Bank competition in new funding to firms

In this Appendix, we show that for each \( \rho \in [0, \bar{\rho}] \) the competitive bank new overall debt promise \( b'_N = b^*_{N}(\rho) \) at \( t = 1 \) against \( \rho \) units of funds to type-\( \rho \) firms at \( t = 0 \) defined in Section 2 as the minimum solution \( b_N \) to (4) and (5), would arise as a result of Bertrand competition in new funding to the firms between the bank and a potential new bank entrant.

Suppose that there is a second bank, the entrant, that only differs from the one presented in Section 2, the incumbent, in that it does not have any assets nor liabilities at \( t = 0 \).

At \( t = 0 \), the incumbent and the entrant set simultaneously the loan promises \( b_L \) and \( \tilde{b}_L \), respectively, they require in exchange of \( \rho \) units of funds to type-\( \rho \) firms at \( t = 0 \). Firms choose the lowest offer, and in case of indifference we adopt the convention that they choose that of the incumbent. The overall firm's debt is hence \( b_O + \min (b_L, \tilde{b}_L) \). We assume that both the owner of the incumbent and its existing debtors (who are infinitely risk-averse investors) have the option to liquidate the firms whenever that is in their interest given the outcome of the competition in new financing. To avoid firms' liquidation by these two agents when the entrant wins the new financing offer, we assume that, in addition to the loan offer \( \tilde{b}_L \), the entrant \( i) \) issues a guarantee on the incumbent’s debt that is described by some \( \theta \)-contingent transfers \( \epsilon(\theta | \tilde{b}_L) \equiv (d_O - \theta \hat{\theta} (b_O + \tilde{b}_L) b_O) - \tilde{b}_L - \rho, \) at \( t = 1 \) so that their debt remain safe; and \( ii) \) makes a defaultable promise \( d'_I \) at \( t = 1 \) to the incumbent’s owner. We assume that the repayment of the guarantee \( \epsilon(\theta) \) is junior to that of the entrant’s own debtors, and that the repayment of \( d'_I \) is junior to that of \( \epsilon(\theta) \).

A funding offer by the entrant is thus described by a tuple \( (\tilde{b}_L, \epsilon(\theta | \tilde{b}_L), d'_I) \). The tuple is feasible whenever, if chosen by the firms, it allows the entrant to obtain new funding from savers and to satisfy the guarantee on the incumbent’ debtors, and does not lead the incumbent’s owner to exercise its liquidation option. The tuple is thus feasible if the following three conditions are met:

- The entrant’s debt are safe:
  \[ \rho \leq \theta \hat{\theta} (b_O + \tilde{b}_L) \tilde{b}_L. \] (44)

- The entrant is able to satisfy the guarantee on the incumbent’s debt for any \( \theta \):
  \[ \epsilon(\theta | \tilde{b}_L) = (d_O - \theta \hat{\theta} (b_O + \tilde{b}_L) b_O) + \leq \theta \hat{\theta} (b_O + \tilde{b}_L) \tilde{b}_L - \rho, \] (45)

  where the RHS captures the entrant’s available payoffs after repaying its debtors.

- The incumbent’s owner does not find optimal to liquidate the firms:
  \[ E \left[ \left( \theta \hat{\theta} (b_O + \tilde{b}_L) b_O + \min \left\{ d'_I, (\theta \hat{\theta} (b_O + \tilde{b}_L) b_O - \rho - \epsilon(\theta | \tilde{b}_L)) \right\} - d_O \right) \right] \geq R - d_O, \] (46)

  where the LHS captures the utility the incumbent’s owner obtains from the residual claim on its bank and the promise \( d'_I \) made by the entrant, and the RHS captures its utility under liquidation.
The funding offer by the incumbent is described by the promise \( b_L \) and, as discussed in Section 2, is feasible if \( b_N = b_O + b_L \) satisfies (4) and (5).

We can without loss of generality focus on funding offers by the two banks that are feasible.

We have the following result.

**Lemma.** The set of \( b_L \geq 0 \) such that there exist \( d'_i \geq 0 \) such that the entrant’s funding offer tuple \( (\bar{b}_L, \epsilon(\theta|\bar{b}_L), d'_i) \) is feasible coincides with the set \( b_L \geq 0 \) such that the promise \( b_L \) is feasible by the incumbent.

**Proof.** Suppose \( b_L > 0 \) is a feasible promise by the incumbent, that is, a promise such that \( b_N = b_O + b_L \) satisfies (4) and (5). Define \( \bar{b}_L = b_L \). We have that

\[
d_O = \theta \bar{p} (b_O) b_O > \theta \bar{p} (b_O + \bar{b}_L) b_O,
\]

and then (4) implies that (44) is strictly satisfied. Moreover, (4) implies that (45) is satisfied for \( \theta = \theta \), and this in turn implies that (45) is satisfied for any \( \theta \).

We have by construction that:

\[
(\theta \bar{p} (b_O + \bar{b}_L) b_O - d_O)^+ + (\theta \bar{p} (b_O + \bar{b}_L) \bar{b}_L - \rho - \epsilon(\theta|\bar{b}_L))^+ = \theta \bar{p} (b_O + \bar{b}_L) (b_O + \bar{b}_L) - d_O - \rho, \tag{47}
\]

and then (5) implies that for a sufficiently large \( d'_i \) the condition (46) will be satisfied.

Suppose now that \( (\bar{b}_L, \epsilon(\theta|\bar{b}_L), d'_i) \) satisfies (44)-(46). Define \( b_L = \bar{b}_L \) and \( b_N = b_O + b_L \). We have that (44) and (45) imply that (4) is satisfied. In addition, using (47) we have that:

\[
\bar{p} (b_N) b_N - d_O - \rho = E \left[ (\theta \bar{p} (b_O + \bar{b}_L) b_O + (\theta \bar{p} (b_O + \bar{b}_L) \bar{b}_L - \rho - \epsilon(\theta|\bar{b}_L))^+ - d_O) \right] \geq \geq E \left[ (\theta \bar{p} (b_O + \bar{b}_L) b_O + \min \{d'_i, (\theta \bar{p} (b_O + \bar{b}_L) \bar{b}_L - \rho - \epsilon(\theta|\bar{b}_L))^+ \} - d_O) \right] \geq R - d_O,
\]

where in the second inequality we have used (46). We have thus that (5) is satisfied.

The lemma states that the incumbent and the entrant can make the same feasible offers to the firms, which will then chose the lowest one. Bertrand competition then immediately implies that in equilibrium the incumbent will set its minimum feasible promise, as stated in Section 2.

### A.2 Proofs of Lemmas and Propositions

**Proof.** Lemma 1.

We split the proof in a sequence of steps.

i) From Assumption 1, we have for \( b \in [0, \eta] \), \( A - b \geq e'(\bar{p}) \), so problem (3) is solved at the corner \( p = \bar{p} \). While for \( b \in [\eta, A] \), the FOC implies \( e'(\bar{p}(b)) = A - b \).

ii) From i) and Assumption 1, we have: for \( b \in [\eta, A] \), \( \bar{p}'(b) = -1/e''(\bar{p}(b)) < 0 \), \( \bar{p}(\eta) = \bar{p} \), and \( \bar{p}(A) = 0 \).
iii) From ii) and Assumption 1, we have that \( \tilde{p}(b)b \) is concave in \( b \) and equals zero at \( b = 0 \) and at \( b = A \). And, it is increasing at \( b = \eta \). So, it has a maximum at \( b_{\max} \in (\eta, A) \).

\( \square \)

**Proof. Lemma 2.**

We proceed with the proof in a sequence of steps. All statements are meant to hold almost surely, that is, except for sets with zero measure. The same applies for the remaining proofs in this appendix. The overall intuition behind formal arguments in this proof is that, for a given continuation and liquidation of firms, consumption redistribution across agents is welfare neutral unless when it affects the effort of firms that continue or the government’s debt in excess of its fiscal slack.

i) \( \Gamma^*_C \) has positive measure.

It is immediate from Assumptions 6 - 4 and the welfare expression in (19) that the allocation induced in the no-intervention benchmark described in Section 3 would be feasible, exhibits the continuation of firms with \( \rho \) sufficiently close to zero, and leads to more welfare than the liquidation of all firms.

ii) \( x^*(\theta) > X \) for all \( \theta \).

Suppose to the contrary that \( x^*(\theta) \leq X \) for \( \theta \in \Omega \) where \( \Omega \) has positive measure.

Case I. \( p^*(\rho) < \overline{\rho} \) for \( \rho \in \Delta \) where \( \Delta \subset \Gamma^*_C \) has positive measure.

Consider the following alternative allocation \( \Lambda' \) which differs from the optimal \( \Lambda^* \) in: i) \( x'(\theta) = x^*(\theta) + \epsilon \) for all \( \theta \in \Omega \) for some small \( \epsilon > 0 \); ii) \( c'_{F,C}(A, \theta, \rho) = c'_{F,C}(A, \theta, p) + e'(\theta) \) where \( e'(\theta) = \epsilon / \int_{\Delta} p^*(\rho) d\rho \) for \( \theta \in \Omega, \rho \in \Delta \); iii) \( p'(\rho) \) for \( \rho \in \Delta \) is defined by (17) given the change in the consumption to firms described in ii), which implies that \( p'(\rho) > p^*(\rho) \); iv) \( c'_B(\rho, \theta) \) for \( \theta \in \Omega, \rho \in \Delta \); is defined by (13) given the allocation changes defined in i)-iii), which implies that \( c'_B(\rho, \theta) > c'_B(\rho, \theta) \). By construction we have that the allocation \( \Lambda' \) is feasible. In addition, since \( \epsilon \) is small, \( g'(0) = 0, x^*(\theta) \leq X \) for \( \theta \in \Omega \), and \( p'(\rho) > p^*(\rho) \) for \( \rho \in \Delta \), we have from (19) that \( Y(\Lambda') > Y(\Lambda^*) \).

Case II. \( p^*(\rho) = \overline{\rho} \) for all \( \rho \in \Gamma^*_C \neq [0, \overline{\rho}] \).

Let us distinguish two sub-cases. First, suppose that \( x^*(\theta) \leq X \) for all \( \theta \). Consider the following alternative \( \Lambda' \) which differs from the optimal \( \Lambda^* \) in: i) \( x'(\theta) = x^*(\theta) + \epsilon \) for all \( \theta \) for some small \( \epsilon > 0 \); ii) \( \Gamma^*_C = \Gamma^*_C \cup \Theta \) where \( \Theta \cap \Gamma^*_C = \emptyset \) and \( \int_{\Theta} \rho d\rho = \epsilon \); iii) \( c'_{F,C}(A, \theta, \rho) = A - b_{O}, c'_{F,C}(0, \theta, \rho) = 0, c'_B(\theta, \rho) = \theta b_o - d_{O} \) for all \( \theta \) and \( \rho \in \Theta \). By construction we have that the allocation \( \Lambda' \) is feasible. In addition, since \( \epsilon \) is small, \( g'(0) = 0, x^*(\theta) \leq X \) for all \( \theta \), \( \overline{\rho} A - \epsilon(\overline{\rho}) > \rho \) for \( \rho \in \Theta \) from Assumption 4, we have from (19) that \( Y(\Lambda') > Y(\Lambda^*) \).

Second, suppose that \( x^*(\theta) > X \) for \( \theta \in \tilde{\Theta} \) where \( \tilde{\Theta} \) has positive measure. We have from the allocation feasibility conditions that \( p^*(\rho) = \overline{\rho} \) for all \( \rho \in \Gamma^*_C \) requires that \( E[x^*(\theta)] \geq \int_{\Gamma^*_C} \rho d\rho \). Consider now the alternative allocation \( \Lambda' \) described as: i) \( \Gamma^*_C = \Gamma^*_C \); ii) \( x'(\theta) = \int_{\Gamma^*_C} \rho d\rho \) for all \( \theta \); iii) the remaining consumption allocations as those induced in an economy with no-lockdown for \( \rho \in \Gamma^*_C \), and as those induced by liquidation for \( \rho \notin \Gamma^*_C \). By construction we have that the allocation \( \Lambda' \) is feasible, induces effort \( \overline{\rho} \) for \( \rho \in \Gamma^*_C = \Gamma^*_C \), and satisfies \( E[x'(\theta)] \leq E[x^*(\theta)] \). Taking into account that \( x'(\theta) \) is constant while \( x^*(\theta) \) is strictly above and weakly below \( X \) in sets with positive measure, that \( g(.) \) is convex, we conclude from (19) that \( Y(\Lambda') > Y(\Lambda^*) \).

Case III. \( p^*(\rho) = \overline{\rho} \) for all \( \rho \in \Gamma^*_C = [0, \overline{\rho}] \).
The feasibility conditions require that $E[x^*(\theta)] \geq \int \rho d\rho > X$, where in the last inequality we have used Assumption 7. We necessarily have that $x^*(\theta) > X$ for $\theta \in \bar{\Omega}$ where $\bar{\Omega}$ has positive measure. Then the arguments in the second case of Part II can be reproduced.

iii) $c^*_F(0,\theta,\rho) = 0$ for all $\rho \in \Gamma_\Delta$ and $c^*_F(\theta,\rho) = 0$ for all $\rho \notin \Gamma_\Delta$ and for all $\theta$.

Suppose $c^*_F(0,\theta,\rho) > 0$ for $\theta \in \Delta, \rho \in \Delta \subset \Gamma_\Delta$ where $\Delta, \Delta$ have positive measure. By choosing $\epsilon > 0$ sufficiently small and suitably reducing the sets $\Delta, \Delta$, we can assume that $c^*_F(0,\theta,\rho) > \epsilon$ for $\theta \in \Delta, \rho \in \Delta \subset \Gamma^*_\Delta$. Consider the following alternative $\Lambda'$ which differs from the optimal $\Lambda^*$ in: i) $c^*_F(0,\theta,\rho) = c^*_F(\Lambda,\theta,\rho) - \epsilon$ for $\theta \in \Delta, \rho \in \Delta$; ii) $x^*(\theta) = x^*(\theta) - \epsilon \int_\Delta d\rho$ for $\theta \in \Omega$. By construction we have that the allocation $\Lambda'$ is feasible, induces effort $p'(\rho) \geq p^*(\rho)$ for $\rho \in \Delta$. Since $x^*(\theta) > X$ for all $\theta$ from ii) and $x'(\theta) < x^*(\theta)$ for $\theta \in \Omega$ we conclude from (19) that $Y(\Lambda') > Y(\Lambda^*)$.

The other statement in iii can be proved analogously.

iv) Constraints in (14), (15) and (16) are binding.

The proof is analogous to that of iii).

v) $X^* \equiv E[x^*(\theta)] \leq \int_{\rho \in \Gamma_\Delta} \rho h(\rho) d\rho$.

Suppose that $X^* > \int_{\rho \in \Gamma^*_\Delta} \rho h(\rho) d\rho$. The feasible allocation $\Lambda'$ constructed in the second part of Case II satisfies $p'(\rho) = \tilde{p} \geq p^*(\rho)$ for $\rho \in \Gamma^*_\Delta = \Gamma^*_\Delta$, and $E[x'(\theta)] < E[x^*(\theta)]$. The same arguments those done before then imply that $Y(\Lambda') > Y(\Lambda^*)$.

\[ \square \]

**Proof.** Lemma 3.

Recall the optimality properties in Lemma 2 that an optimal $\Lambda^*$ must satisfy. Given the definition of $b^*(\rho)$ in (22), the allocation $\Lambda^*$ also satisfies the conditions derived in the main text (23) and (24). It suffices to prove the following statements.

i) $\Gamma^*_\Delta = [0,\rho^*]$ for $\rho^* > 0$.

Suppose the statement is not true. There exist then $\rho_1 > \rho_2$ with $\rho_1 \in \Gamma^*_\Delta, \rho_2 \notin \Gamma^*_\Delta$. Suppose to fix ideas that $h(\rho_1) > h(\rho_2)$. Consider the following marginal change $\Lambda'$ in the optimal allocation $\Lambda^*$: i) firm-types $\rho_2$ continue with consumption $c^*_F(z,\theta,\rho_2) = c^*_F(z,\theta,\rho_1)$ for $z = A, 0$ and all $\theta$, and their bank consumption is $c^*_b(\theta,\rho_2) = c^*_b(\theta,\rho_1) + \rho_1 - \rho_2$; ii) a fraction $h(\rho_2)/h(\rho_1)$ of firm-types $\rho_1$ gets liquidated with consumption $c^*_F(\theta,\phi(\rho)) = 0$, and their bank consumption is $c^*_b(\theta,\rho_1) = c^*_b(\theta,\rho_2) = R - d_0$. By construction we have that the marginal change in $\Lambda'$ induces effort $p'(\rho_2) = p^*(\rho_1)$ and thus leads to a marginal change in welfare of $dY(\Lambda^*) = (\rho_1 - \rho_2)h(\rho_2) d\rho > 0$. The case $h(\rho_1) < h(\rho_2)$ can be proved analogously.

ii) $\tilde{p}(\theta)$ and $\tilde{p}(\theta)A - \epsilon(\tilde{p}(\theta))$ are concave in $b$.

Direct consequence of Assumption 1 and Lemma 1.

iii) For all $\rho \in \Gamma^*_\Delta$ we have $b^*(\rho) = b^*$.

Suppose there exist $\rho_1, \rho_2 \in \Gamma^*_\Delta$ with $b_1 = b^*(\rho_1) < b_2 = b^*(\rho_2)$. Denote $\tilde{b} = 1/2(b_1 + b_2)$ and $\tilde{c}_{F,C}(\theta) = 1/2(c^*_F(A,\theta,\rho_1) + c^*_F(A,\theta,\rho_2))$. By construction we have $\tilde{b} = A - E[\theta \tilde{c}_{F,C}(\theta)]$. Suppose to fix ideas that $h(\rho_1) > h(\rho_2)$. Consider the following marginal change $\Lambda'$ in the optimal allocation $\Lambda^*$: i) for firm-types $\rho_2$
and for a fraction $h(p_2)/h(p_1)$ of firm-types $\rho_1$ set $c^{\prime}_{F,C}(A,\theta,\rho) = c_{F,C}^{\prime}(\theta)$, which induces effort $p^\prime(\rho) = \hat{p}(\hat{b})$; ii) $x^\prime(\theta) - x^* \geq 0$ for all $\theta$. By construction, the marginal change $\Lambda'$ in the optimal allocation $\Lambda^*$ is feasible. Notice in particular that the $\theta$-contingent redistribution of consumption between firm-types $\rho_1, \rho_2$ (which leaves their overall consumption constant) leads from the concavity of $\hat{p}(b)$ to an increase in $\theta$-contingent firms' output, which we entirely use to marginally reduce government debt. Using also that $\hat{p}(b) \Lambda - e(\hat{p}(b))$ is concave, we conclude from (19) that the marginal change in $\Lambda^*$ leads to a marginal increase in welfare $\Lambda^* > 0$.

**Proof.** Lemma 4.

Recall the optimality properties in Lemma 2 and 3 that an optimal $\Lambda^*$ must satisfy. In particular we have that $c_1^* = (d_0 + E[\rho|\rho \leq \rho^*]) H(\rho^*)$. For given $\hat{x}$, denote $F(\theta|\hat{x}) = \max \{ \hat{x}, c_1^* - \theta p^* AH(\rho^*) \}$. We prove the Lemma in a sequence of steps.

i) There exist unique $\hat{x}^*$ such that $E[F(\theta|\hat{x})] = \hat{x}^*$, and satisfies $\hat{x}^* \leq x^*$ with equality iff (28) is satisfied.

The identity (25) implies $c_1^* - \theta p^* AH(\rho^*) < \hat{x}^*$ for $\theta \geq 1$. Using this inequality the claim results from the following observations: i) $F(\theta|\hat{x}) = \hat{x}^*$ for all $\theta$ and with equality iff (28) is satisfied; ii) $E[F(\theta|\hat{x})]$ is increasing in $\hat{x}$ for $\hat{x} \leq \hat{x}^*$ and, taking into account that $1/p$ is the maximum value of $\theta$, strictly so if $\hat{x} \geq c_1^* - 1/\bar{p} p^* AH(\rho^*)$; iii) $E[F(\theta|\hat{x})] = c_1^* - p^* AH(\rho^*) < \hat{x}^*$ for $\hat{x} < c_1^* - 1/\bar{p} p^* AH(\rho^*)$.

ii) If (28) is satisfied, then $x^*(\theta) = \hat{x}^*$ for all $\theta$.

Suppose the claim is not satisfied. Consider the following alternative $\Lambda'$ which differs from the optimal $\Lambda^*$ in: i) $x^*(\theta) = \hat{x}^*$; ii) $c^{\prime}_{F,C}(A,\theta,\rho) = A - b^*$ for $\rho \in \Gamma^*_C$ and all $\theta$; iii) $c^*_B(\theta,\rho) = \theta p^* b^* - d_0$ for $\rho \in \Gamma^*_C$. By construction, using that (28) is satisfied, we have that the allocation $\Lambda'$ is feasible. (Notice in particular that (13) is satisfied for all $\theta$). Since in some positive measure set $x^*(\theta) > \hat{x}^* > X$, using the convexity of $g(.)$, we conclude from (19) that $\Lambda' > \Lambda^*$.

iii) If (28) is satisfied, then $F(\theta|\hat{x} = \hat{x}^*) = \hat{x}^*$ for all $\theta$.

Immediate from ii) and the definition of $F(\theta|\hat{x})$.

Suppose in the rest of the proof that (28) is not satisfied. Define $\theta^* > 0$ by the property $c_1^* - \theta^* p^* AH(\rho^*) = \hat{x}^*$ where $\hat{x}^* < x^*$ was defined in i).

iv) $x^*(\theta) \geq F(\theta|\hat{x}^*)$ for all $\theta \leq \theta^*$, with equality iff $c_{\hat{x},C}^*(A,\theta,\rho) = c^*_B(\theta,\rho) = 0$ for all $\rho \in [0,\rho^*]$. The claim results immediately from (13), the definitions of $F(\theta|\hat{x}^*)$, $\theta^*$, and the consumption properties in Lemma 2.

v) $x^*(\theta) = F(\theta|\hat{x}^*)$ for all $\theta \leq \theta^*$.

Suppose that $x^*(\theta) > F(\theta|\hat{x}^*)$ for $\theta \in \Omega \subset [\theta^*, \theta^*]$ where $\Omega$ has positive measure. Since $E[x^*(\theta)] = E[F(\theta|\hat{x}^*)]$ then using iv) we must have that $x^*(\theta) < F(\theta|\hat{x}^*)$ for $\theta \in \Omega' \subset [\theta^*/1/\bar{p}]$ where $\Omega'$ has positive measure. From the last part of iv) we can distinguish two cases:

Case I: $c_{F,C}^*(A,\theta,\rho) > 0$ for $\theta \in \Omega, \rho \in \Delta$ where $\Delta \subset [0,\rho^*]$ has positive measure.

\begin{align*}
&\text{Case I: } c_{F,C}^*(A,\theta,\rho) > 0 \quad \text{for } \theta \in \Omega, \rho \in \Delta \quad \text{where } \Delta \subset [0,\rho^*] \quad \text{has positive measure.}
\end{align*}
Consider the following alternative $\Lambda'$ which differs from the optimal $\Lambda^*$ in: i) $c'_{f,c}(A, \theta, \rho) = c^*_f(A, \theta, \rho) - \epsilon$ for $\theta \in \Omega, \rho \in \Delta$, where $\epsilon$ is sufficiently small; ii) $c'_{f,c}(A, \theta, \rho) = c^*_f(A, \theta, \rho) + \epsilon'$ for $\theta \in \Omega', \rho \in \Delta$, where:

$$
\epsilon' = \frac{\int_\Omega \theta d\theta}{\int_{\Omega'} \theta d\theta} \epsilon_i
$$

(48)

iii) $x'(\theta) = x^*(\theta) - \int_\Delta \theta p^*(\rho) e d\rho$ for $\theta \in \Omega$, $x'(\theta) = x^*(\theta) + \int_\Delta \theta p^*(\rho) \epsilon' d\rho$ for $\theta \in \Omega'$.

We have using (48) and (17) that $\Lambda'$ induces $p^*(\rho)$ for $\rho \in \Delta$, so that the alternative allocation is feasible. In addition, it satisfies also from (48):

$$
\int_{\Omega \cup \Omega'} x'(\theta) = \int_{\Omega \cup \Omega'} x^*(\theta) d\theta.
$$

Using that $x^*(\theta) > X$ for all $\theta$ from Lemma 2, we have from the equality above and the convexity of $g(.)$ that

$$
\int_{\Omega \cup \Omega'} g \left( (x^*(\theta) - X)^+ \right) d\theta > \int_{\Omega \cup \Omega'} g \left( (x'(\theta) - X)^+ \right) d\theta,
$$

and we conclude from (19) that $Y(\Lambda') > Y(\Lambda^*)$.

Case II: $\hat{c}_b^*(\theta, \rho) > 0$ for $\theta \in \Omega, \rho \in \Delta$ where $\Delta \subset [0, \rho^*]$ has positive measure.

The proof is analogous to that of Case I.

vi) $x^*(\theta) = F(\theta | \hat{x}^*)$ for all $\theta$.

Suppose that $x^*(\theta) > F(\theta | \hat{x}^*)$ for some $\theta > \theta^*$. Then by the definition of $\theta^*$ we have that either $c^*_f(A, \theta, \rho) > 0$ or $c^*_b(\theta, \rho) > 0$ for $\rho \in \Delta$ where $\Delta$ has positive measure. Taking this observation into account, the arguments in vi) can be reproduced to prove the claim.

\[ \square \]

Proof. Proposition 1.

For notational convenience we drop the explicit dependence of optimal variables on $X$ from most expressions, unless necessary for clarifications.

From the derivations in Section 4.2 and Lemmas 2-4, we have that the SP optimal allocations are described by $(p^*, \rho^*, \hat{x}^*)$ that maximize the following SP objective (re-written from 19):

$$
\max_{(p^* \in [0, \bar{p}], \rho^* \in [0, \bar{\rho}], \hat{x}^* \in [X, \bar{x}, \rho h(\rho) d\rho]} \int_0^{\rho^*} \left[ (p^* A - e(p^*) - \rho) \right] h(\rho) d\rho + \int_{\rho > \rho^*} R h(\rho) d\rho - C(x^*(\theta | p^*, \rho^*, \hat{x}^*), p^*, \rho^*),
$$

(49)

subject to the funding constraint:

$$
\int_0^{\rho^*} [R + \rho] h(\rho) d\rho = p^* b(p^*) H(\rho^*) + \hat{x}^*,
$$

(50)
where $x^*(\theta|p^*,\rho^*,x^*)$ is defined in Lemma 4 and $C(x^*(\theta),p^*,\rho^*) = E[g(x^*(\theta) - X^+)]$.

Concavity of $\hat{p}(b)A - e(\hat{p}(b))$ (see proof of Lemma 3) and Assumption 6 imply that the objective function is strictly concave, and Lemma 1 implies that the choice set is compact, then there exists a unique solution $(p^*,\rho^*,\bar{x}^*)$.

We prove sequentially the two statements in the proposition.

i) $\rho^*(X), p^*(X), \bar{x}^*(X)$ are increasing in $X$, with $d\bar{x}^*(X)/dX < 1$.

We distinguish two cases:

I-Safe collateral abundance. Consider some $X$ such that $(p^*,\rho^*,x^*)$ satisfies (28). We have that $x^*(\theta) = x^*$ for all $\theta$, and $C(p^*,\rho^*,x^*) = g(\bar{x}^* - X)$. Let $\lambda$ be the Lagrange multiplier associated with constraint (50). The FOCs of the SP problem are:

- With respect to $x^*$:
  \[ \lambda = g'(\bar{x}^* - X) \] (51)

- With respect to $p^*$:
  \[ (A - e'(p^*)) - \lambda \left( - \frac{\partial \hat{p}(p)}{\partial p} \right) = 0, \]
  which after replacing $\lambda$ leads to
  \[ (A - e'(p^*)) = g'(\bar{x}^* - X) \left( - \frac{\partial \hat{p}(p)}{\partial p} \right). \] (52)

- With respect to $\rho^*$:
  \[ (p^* A - e(p^*) - \rho^*) - R - \lambda ([R + \rho^*] - p^*b(p^*)) = 0, \]
  which after replacing $\lambda$ leads to
  \[ p^* A - e(p^*) - \rho^* - R = g'(\bar{x}^* - X) (R + \rho^* - p^*b(p^*)). \] (53)

The SP solution $(p^*,\rho^*,\bar{x}^*)$ is determined by equations (52), (53) and (50). Taking implicit derivatives with respect to $X$ to this system, we get:

- Implicit derivative of (52) with respect to $X$:
  \[ \frac{\partial \Omega(p^*)}{\partial p^*} \frac{\partial p^*(X)}{\partial X} = -g''(\bar{x}^* - X) \left( 1 - \frac{\partial \bar{x}^*(X)}{\partial X} \right). \] (54)

where we define $\Omega(p) \equiv \frac{A - e'(p)}{p^* - e'(p)} > 0$ and $\frac{\partial \Omega(p)}{\partial p} = -\frac{e''(p)}{pc''(p) - b(p)} - \Omega(p) \frac{2e''(p) + pc''(p)}{pc''(p) - b(p)} < 0.$
• Implicit derivative of (53) with respect to $X$:

$$
(1 + g'(x^* - X)) \frac{\partial p^*(X)}{\partial X} = g''(x^* - X) \left( 1 - \frac{\partial \bar{x}^*(X)}{\partial X} \right) ([R + \rho^*] - p^*b(p^*)).
$$

\hspace{2cm} (55)

• Implicit derivative of (50) with respect to $X$:

$$
\left( R + \rho - p^* \hat{b}(p^*) \right) h(\rho^*) \frac{\partial p^*(X)}{\partial X} = \frac{\partial p^* \hat{b}(p^*)}{\partial p^*} H(\rho^*) \frac{\partial p^*(X)}{\partial X} + \frac{\partial \bar{x}^*(X)}{\partial X}.
$$

\hspace{2cm} (56)

We can see from (54) and (55) that $\frac{\partial p^*(X)}{\partial X} > 0$ and $\frac{\partial \bar{x}^*(X)}{\partial X} > 0$ if and only if $\frac{\partial \bar{x}^*(X)}{\partial X} < 1$.

We next prove that $\frac{\partial \bar{x}^*(X)}{\partial X} \in (0, 1)$. Plugging $\frac{\partial p^*(X)}{\partial X}$ and $\frac{\partial \bar{x}^*(X)}{\partial X}$ from (54) and (55) into (56), after some manipulations, we get:

$$
\left[ \left( \frac{([R + \rho^*] - p^*b(p^*))^2}{1 + g'(x^* - X)} \right) h(\rho^*) + \frac{-\partial p^* \hat{b}(p^*)}{\partial p^*} \left( -\frac{\partial \Omega(p^*)}{\partial p^*} \right)^{-1} H(\rho^*) \right] g''(x^* - X) = \frac{\frac{\partial \bar{x}^*(X)}{\partial X}}{1 - \frac{\partial \bar{x}^*(X)}{\partial X}}.
$$

\hspace{2cm} (57)

Denoting $m_p \equiv \left( \frac{([R + \rho^*] - p^*b(p^*))^2}{1 + g'(x^* - X)} \right) h(\rho^*)$ and $m_\rho \equiv -\frac{\partial p^* \hat{b}(p^*)}{\partial p^*} \left( -\frac{\partial \Omega(p^*)}{\partial p^*} \right)^{-1} H(\rho^*)$, we can write

$$
\frac{\partial \bar{x}^*(X)}{\partial X} = \frac{m_p + m_\rho}{g'(x^* - X) + m_p + m_\rho}.
$$

\hspace{2cm} (58)

From Lemma 1 and Assumption 6 we have that $m_p > 0$, $m_\rho > 0$ and $E[g''(x^* - X)] > 0$, and thus $\frac{\partial \bar{x}^*(X)}{\partial X} \in (0, 1)$, which proves our claim.

**II-Safe collateral scarcity.** Consider some $X$ such that $(p^*, \rho^*, x^*)$ does not satisfy (28). We have from Lemma 4 that $x^*(\theta) = \hat{x}^* < x^*$ for $\theta > \theta^*$ while $x^*(\theta) > \hat{x}^*$ is decreasing in $\theta$ for $\theta < \theta^*$. We thus have:

$$
C(p^*, \rho^*, x^*) = \int_{\theta}^{\theta^*} g (x^*(\theta) - X) dG(\theta) + g (\hat{x}^* - X) (1 - G(\theta^*)) ,
$$

and we can write

$$
x^*(\theta) = \Delta_1(\rho^*) - \min \{ \theta, \theta^* \} \Delta_2(p^*, \rho^*)
$$

\hspace{2cm} (59)

with

$$
\Delta_1(\rho^*) \equiv (d_0 + E[\rho(\rho < \rho^*)]) H(\rho^*), \Delta_2(p^*, \rho^*) \equiv p^* \Lambda H(\rho^*),
$$

57
and where \( \theta^* \) is a function of \((p^*, \rho^*, x^*)\) defined by:

\[
\int_{\tilde{\theta}}^{\theta^*} x(\theta) dG(\theta) + \int_{\theta^*}^{\infty} x(\theta^*) dG(\theta) = x^*.
\] (60)

The FOCs of the SP problem in this case can be written as:

- With respect to \( x^* \):
  \[
  \lambda = \frac{\partial C(p^*, \rho^*, x^*)}{\partial x^*} = g'(x(\theta^*) - X).
\] (61)

- With respect to \( p^* \):
  \[
  (A - e'(p^*)) - \lambda \left( -\frac{\partial \hat{b}(p)}{\partial p} \right) = \frac{\partial C(p^*, \rho^*, x^*)}{\partial p^*},
\]
  where the new term in the RHS appears due to the direct effect of \( p^* \) on the deadweight cost of public debt when there is safe collateral scarcity. After some manipulations we can write

\[
\frac{\partial C(p^*, \rho^*, x^*)}{\partial p^*} = -AH(p^*) \left\{ \int_{\tilde{\theta}}^{\theta^*} \left( g' \left( x(\theta) - X \right) - g' \left( x(\theta^*) - X \right) \right) (\theta) dG(\theta) \right\} AH(p^*) < 0.
\]

After replacing \( \lambda \), we get:

\[
\Omega(p^*) = g' \left( x(\theta^*) - X \right) + \Psi_p(p^*, \rho^*, x^*) = \frac{\partial C(p^*, \rho^*, x^*)}{\partial p^*} \left( -\frac{\partial \hat{b}(p)}{\partial p^*} \right).
\] (62)

- With respect to \( \rho^* \):
  \[
  (p^* A - e(p^*) - \rho^* - R) h(\rho^*) - \lambda (R + \rho^* - p^* b(p^*)) h(\rho^*) = \frac{\partial C(p^*, \rho^*, x^*)}{\partial \rho^*},
\]
  where the new term in the RHS appears due to the direct effect of \( \rho^* \) on the deadweight cost of public debt when there is safe collateral scarcity. After some manipulations we can write

\[
\frac{\partial C(p^*, \rho^*, x^*)}{\partial \rho^*} = \left\{ \int_{\tilde{\theta}}^{\theta^*} \left( g' \left( x(\theta) - X \right) - g' \left( x(\theta^*) - X \right) \right) ((d_0 + \rho^*) h(\rho^*) - \theta Ap^* h(\rho^*)) dG(\theta) \right\} > 0.
\]
Both expressions (65) and (67) depend on $\frac{dx}{X}$ derivatives with respect to $X$. In this case, the SP solution $(p^*, \rho^*, \bar{x}^*)$ is determined by equations (62), (63) and constraint (50). Taking implicit derivatives with respect to $X$ to this system, we get:

- Implicit derivative of (62) with respect to $X$:

$$\frac{\partial \Omega(p^*)}{\partial p^*} \frac{\partial p^*(X)}{\partial X} = -g''(x(\theta^*) - X) \left( 1 - \frac{dx(\theta^*)}{dX} \right),$$

and we can write:

$$\frac{\partial p^*(X)}{\partial X} = g''(x(\theta^*) - X) \left( 1 - \frac{dx(\theta^*)}{dX} \right) \frac{1}{-\frac{\partial \Omega(p^*)}{\partial p^*}}.$$  

(65)

- Implicit derivative of (63) with respect to $X$:

$$(1 + g'(x(\theta^*) - X)) \frac{\partial \rho^*(X)}{\partial X} = g''(x(\theta^*) - X) \left( 1 - \frac{dx(\theta^*)}{dX} \right) (R + \rho^*) - p^* b(p^*),$$

and we can write:

$$\frac{\partial \rho^*(X)}{\partial X} = \frac{g''(x(\theta^*) - X)}{(1 + g'(x(\theta^*) - X))} \left( 1 - \frac{dx(\theta^*)}{dX} \right) (R + \rho^*) - p^* b(p^*).$$

(67)

Both expressions (65) and (67) depend on $\frac{dx(\theta^*)}{dX}$, so we first derive an expression for this object. Note that $x(\theta^*)$ can change directly due to $p^*, \rho^*$ and indirectly due to $\theta^*$ (which in turn changes with $p^*, \rho^*$ and $\bar{x}^*$). Using (59) we have:

$$\frac{dx(\theta^*)}{dX} = \frac{\partial x(\theta^*; p^*, \rho^*, \bar{x}^*)}{\partial \theta^*} \frac{d\theta^*}{dX} + \frac{\partial x(\theta^*; p^*, \rho^*, \theta^*)}{\partial p^*} \frac{dp^*(X)}{dX} + \frac{\partial x(\theta^*; p^*, \rho^*, \theta^*)}{\partial \rho^*} \frac{d\rho^*(X)}{dX} + \frac{\partial x(\theta^*; p^*, \rho^*, \bar{x}^*)}{\partial \bar{x}^*} \frac{d\bar{x}^*(X)}{dX}.$$

Using (60), it follows that:

$$\frac{d\theta^*}{dX} = \frac{\partial \theta^*(p^*, \rho^*, \bar{x}^*)}{\partial p^*} \frac{dp^*(X)}{dX} + \frac{\partial \theta^*(p^*, \rho^*, \bar{x}^*)}{\partial \rho^*} \frac{d\rho^*(X)}{dX} + \frac{\partial \theta^*(p^*, \rho^*, \bar{x}^*)}{\partial \bar{x}^*} \frac{d\bar{x}^*(X)}{dX}.$$
Putting the above equations together, we can write:

\[
\frac{\partial \theta^*(p^*, \rho^*, \bar{x}^*)}{\partial p^*} = -\frac{A \left( \int_\theta^s \theta dG(\theta) + \theta^* (1 - G(\theta^*)) \right)}{\Delta_2(p^*, \rho^*) (1 - G(\theta^*))} H(\rho^*),
\]

\[
\frac{\partial \theta^*(p^*, \rho^*, \bar{x}^*)}{\partial \rho^*} = \frac{(d_0 + \rho^*) - Ap^* \left( \int_\theta^s \theta dG(\theta) + \theta^* (1 - G(\theta^*)) \right)}{\Delta_2(p^*, \rho^*) (1 - G(\theta^*))} h(\rho^*),
\]

\[
\frac{\partial \theta^*(p^*, \rho^*, \bar{x}^*)}{\partial x^*} = -\frac{1}{\Delta_2(p^*, \rho^*) (1 - G(\theta^*))}.
\]

Putting the above equations together, we can write:

\[
\frac{dx(\theta^*)}{dX} = \phi_p \frac{\partial p^*(X)}{\partial X} - \phi_{\rho} \frac{\partial \rho^*(X)}{\partial X} + \phi_x \frac{\partial x^*(X)}{\partial X}, \tag{68}
\]

\[
\phi_p = AH(\rho^*) E[\theta|\theta < \theta^*] \frac{G(\theta^*)}{1 - G(\theta^*)} > 0, \tag{69}
\]

\[
\phi_{\rho} = \frac{G(\theta^*)}{1 - G(\theta^*)} \left( (d_0 + \rho^*) - Ap^* E[\theta|\theta < \theta^*] \right) h(\rho^*) > 0, \tag{70}
\]

\[
\phi_x = \frac{1}{(1 - G(\theta^*))} > 1. \tag{71}
\]

Replacing the expression (68) into (65) and (67), we get:

\[
\left( -\frac{\partial \Omega(p^*)}{\partial p^*} + g''(x(\theta^*) - X) \phi_p \right) \frac{\partial p^*}{\partial X} = g''(x(\theta^*) - X) \left( 1 - \phi_x \frac{\partial x^*}{\partial X} \right) + g''(x(\theta^*) - X) \phi_p \frac{\partial \rho^*}{\partial X},
\]

and

\[
\left( \frac{(1 + g'(x(\theta^*) - X))}{[R + \rho^*] - p^*b(p^*)} - g''(x(\theta^*) - X) \phi_p \right) \frac{\partial \rho^*}{\partial X} = g''(x(\theta^*) - X) \left( 1 - \phi_x \frac{\partial x^*}{\partial X} \right) - g''(x(\theta^*) - X) \phi_p \frac{\partial \rho^*}{\partial X}.
\]

After combining these two equations we get:

\[
\left( \frac{1}{g''(x(\theta^*) - X)} - \phi_p \frac{[R + \rho^*] - p^*b(p^*)}{(1 + g'(x(\theta^*) - X))} + \phi_p \left( -\frac{\partial \Omega(p^*)}{\partial p^*} \right)^{-1} \right) \frac{\partial p^*}{\partial X} = \left( 1 - \phi_x \frac{\partial x^*}{\partial X} \right) \left( -\frac{\partial \Omega(\rho^*)}{\partial \rho^*} \right)^{-1}, \tag{72}
\]

60
and

\[
\left( \frac{1}{g''(x(\theta^*)) - X} - \phi_p \left[ R + \rho^* \right] - p^* b(p^*) \right) + \phi_p \left( \frac{-\partial \Omega(p^*)}{\partial p^*} \right)^{-1} \frac{\partial \rho^*}{\partial X} = \left( 1 - \phi_x \frac{\partial \bar{x}^*}{\partial X} \right) \left[ \frac{[R + \rho^*] - p^* b(p^*)}{(1 + g'(x(\theta^*)) - X)} \right]
\]

(73)

Finally, substituting (72) and (73) into (56) we get:

\[
\frac{\partial \bar{x}^*(X)}{\partial X} = \frac{m_p + m_{\rho}}{g''(x - X) + \left[ \phi_x + \frac{\phi_p}{g''(x(\theta^*))} \right] m_{\rho}}.
\]

(74)

Since \( \phi_x > 1, \phi_p > 0, \phi_{\rho} > 0 \) we have immediately that the numerator is positive and the first two terms in the denominator are positive.

We now prove that \( \left[ \phi_x - \frac{\phi_p}{([R + \rho^*] - p^* b(p^*))H(p^*))} \right] > 0 \). Using the definitions of \( \phi_x \) and \( \phi_p \) from (70) and (71), we have that it is the case if and only if

\[
([R + \rho^*] - p^* b(p^*)) - G(\theta^*) \left( (d_0 + \rho^*) - p^* AE[\theta | \theta < \theta^*] \right) > 0.
\]

(75)

Note that from the definition of \( \theta^* \) in (60) and from the funding constraint (50):

\[
\int_{0}^{\theta^*} (\Delta_1(\rho^* - \theta^*) \Delta_2(p^*)) dG(\theta) + \int_{\theta^*}^{0} \Delta_1(\rho^* - \theta^*) \Delta_2(p^*)) g(\theta)d\theta = \bar{x}^* = \int_{\rho_{\theta^*}}^{\rho^*} [R + \rho] h(\rho)d\rho - p^* \hat{b}(p^*) H(p^*),
\]

which implies that \( (d_0 + \rho^*) - p^* A E[\min(\theta, \theta^*)] = [R + \rho^*] - p^* \hat{b}(p^*) \). Then,

\[
([R + \rho^*] - p^* b(p^*)) - G(\theta^*) \left( (d_0 + \rho^*) - p^* AE[\theta | \theta < \theta^*] \right) = (1 - G(\theta^*)) \left( (d_0 + \rho^*) - p^* A \theta^* \right)
\]

\[
> (1 - G(\theta^*)) \left( \int_{0}^{\theta^*} ((d_0 + \rho) dH(\rho^*) - p^* A \theta^* H(\rho^*)) \right) = (1 - G(\theta^*)) x^*(\theta^*) > 0,
\]

(76)

where we have used that, from Lemma 4, \( x^*(\theta^*) > 0 \). Thus, condition (75) holds, and \( \frac{\partial \bar{x}^*(X)}{\partial X} > 0 \).

Now we show that \( \frac{\partial \bar{x}^*(X)}{\partial X} > 0 \) and \( \frac{\partial \bar{x}^*(X)}{\partial \rho} > 0 \). From (74), we have

\[
1 - \phi_x \frac{\partial \bar{x}^*(X)}{\partial X} = \frac{1}{\frac{1}{g''(x - X)} + \phi_p \left( \frac{-\partial \Omega(p^*)}{\partial p^*} \right)^{-1} \frac{\partial \rho^*}{\partial X}} \left[ \phi_x + \frac{\phi_p}{g''(x(\theta^*))} \right] m_{\rho} + \left[ \phi_x - \frac{\phi_p}{([R + \rho^*] - p^* b(p^*))H(p^*)} \right] m_{\rho}
\]

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Then, we can write (72) as:

$$\frac{\partial p^*}{\partial X} \geq \frac{1}{\bar{g}'(x^*-X)} \left[ \phi_x + \frac{\phi_p}{\partial p^*} \left[ \phi_x - \frac{\phi_p}{\partial p^*} \left[ \frac{[R+p^*b(p^*)]}{(1+q'(x^*-X))} \right] m_p + \left[ \phi_x - \frac{\phi_p}{\partial p^*} \left[ \frac{[R+p^*b(p^*)]}{(1+q'(x^*-X))} \right] m_p \right] \right] m_\rho $$

And, we can write (73) as:

$$\frac{\partial \rho^*}{\partial X} = \frac{1}{\bar{g}'(x^*-X)} \left[ \phi_x + \frac{\phi_p}{\partial p^*} \left[ \phi_x - \frac{\phi_p}{\partial p^*} \left[ \frac{[R+p^*b(p^*)]}{(1+q'(x^*-X))} \right] m_p + \left[ \phi_x - \frac{\phi_p}{\partial p^*} \left[ \frac{[R+p^*b(p^*)]}{(1+q'(x^*-X))} \right] m_p \right] \right] m_\rho $$

ii) There exists $X_S \geq 0$ such that there is safe collateral abundance at the optimal allocation if and only if $X \geq X_S$. In addition, $X_S > 0$ if $\theta$ and $\eta$ are sufficiently small.

Let

$$S(p^*, \rho^*, x^*) = x^* - (d_0 + E[\rho | \rho \leq \rho^*]) - \theta p^* A) H(\rho^*).$$

From (28), we have that there is safe collateral abundance iff $S(p^*, \rho^*, x^*) \geq 0$.

From the funding constraint (25), we get (after adding and subtracting $d_0 H(\rho^*)$):

$$x^* = (d_0 + E[\rho | \rho \leq \rho^*]) H(\rho^*) - \left( d_0 + p^* b(p^*) - R \right) H(\rho^*).$$

Then, we can write $S(p^*, \rho^*, x^*)$ as:

$$S^* = \theta p^* A H(\rho^*) - \left( d_0 + p^* b(p^*) - R \right) H(\rho^*).$$

Using Assumptions 3 and 5, we have $S^* > 0$ if and only if

$$\theta > \frac{p^* b(p^*) - \overline{p} b_0}{p^* A - \overline{p} b_0}.$$

Since we have that $p^* b(p^*)$ is decreasing in $X$, while $p^*$ is increasing in $X$, we have that if there is safe collateral abundance for $X'$ then there is safe collateral abundance for any $X > X'$, which proves the first part of statement ii).

The second part results from the following two claims. First, as long as $p^* b(p^*) > \overline{p} b_0$ (which happens when $p^* < \overline{p}$), there exists safe collateral scarcity when $\theta$ is sufficiently small. This results immediately from (79).

Second, if $\eta$ is sufficiently small, there exists $\check{X} > 0$ such that $p^*(X) \hat{b}(p^*(X)) > \overline{p} b_0$ for all $X < \check{X}$. To prove
this claim, we have from the FOCs of the SP problem wrt to \( p^* \) and \( x^* \) that we have the corner solution \( p^* = \bar{p} \) iff

\[
\frac{\hat{b}(\bar{p})}{\hat{b}(\bar{p}) - \frac{\eta}{pe''(\bar{p})}} > g'(\bar{x} - X),
\]

which using Assumption 1 we can write as:

\[
\frac{\eta}{pe''(\bar{p}) - \eta} > g'(\bar{x} - X) > 0.
\]

The LHS is increasing in \( \eta \) while the RHS is decreasing in \( X \), which proves the claim. This concludes our proof of the Proposition.

\[\square\]

**Proof.** Proposition 2.

Suppose the three policies in the proposition have been introduced. We focus the proof in the case in which safe collateral is scarce at the optimal allocation, which is significantly harder to prove. From Lemma 4 we have \( \bar{x} - \hat{x}^* > 0 \) and recall the threshold \( \theta^* > \bar{\theta} \) introduced in that lemma, which using (30) satisfies:

\[
\bar{x} - \hat{x}^* = (\theta^* p^* A - d_O - \tilde{\rho}) H(\rho^*).
\]  

(80)

Given the results in Lemma 5, and our discussion in the main text before the statement of the proposition, it is sufficient to prove the following three claims:

i) \( \tau_B^H(\rho^*) > \bar{x} - \hat{x}^* \)

We have from (27) and (30) the following equality

\[
\bar{x} - x^*(\theta) = \min \{\bar{x} - \hat{x}^*, (\theta p^* A - d_O - \tilde{\rho}) H(\rho^*)\},
\]

and taking expectations we obtain:

\[
E \left[ \min \{\bar{x} - \hat{x}^*, (\theta p^* A - d_O - \tilde{\rho}) H(\rho^*)\} \right] = 0.
\]  

(81)

By definition of fairly priced guarantee, we have that:

\[
E \left[ \min \{\tau_B^l, \theta p^* b_N'(\kappa', \tau_B^l) - d_O - \tilde{\rho}\} \right] = 0.
\]  

(82)

Comparing (81) and (82) and using that \( b_N'(\kappa', \tau_B^l) < A \), we obtain the claim.
ii) If the three policy combination in the Proposition induces the competitive promise $b^*$ for firms that continue, then it also induces $x^*(\theta)$ in (27).

Suppose the policies induce $b^*$ and hence also $p^*$ for firm-types $\rho \leq \rho^*$. Let us distinguish two cases:

Case I: $\theta < \theta^*$

It suffices to prove that $\tau_B' > \theta p^* b^* - d_O - \hat{\rho}$ and $\tau_F(\theta) = 1$. The first claim is true from i), (80), $b^* < A, \theta < \theta^*$. The second claim then results from (39) and the following immediate inequality implied by (80) and $\theta < \theta^*$:

$$\frac{(\bar{x}^* - \hat{x}^*) / H(\rho^*) - (\theta p^* b^* - d_O - \hat{\rho})}{\theta p^*(A - b^*)} > 1.$$  \hspace{1cm} (83)

Case II: $\theta \geq \theta^*$

It suffices to prove that the policy combination induces the government debt target $\hat{x}^*$, which from (39) is the case if inequality (83) does not hold. The later claim is true both when $\tau_B' > \theta p^* b^* - d_O - \hat{\rho}$ (using i)) and when it does not (reproducing the arguments leading to (83)).

iii) If the three policy combination in the Proposition induces the competitive promise $b^*$ for firms that continue, then $\tau_F(\theta)$ in (39) is revenue neutral.

We have from ii) the following equality for all $\theta$:

$$\bar{x}^* - x^*(\theta) = \min \{ \tau_B', \theta p^* b^* - d_O - \hat{\rho} \} + \theta p^* \tau_F(\theta)(A - b^*) \} \cdot H(\rho^*).$$

Taking expectations and using (82), we obtain:

$$0 = E[\theta p^* \tau_F(\theta)(A - b^*)] \cdot H(\rho^*),$$

and $E[\theta \tau_F(\theta)] = 0$. \hfill \qed